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SMARANDACHE NOTIONS

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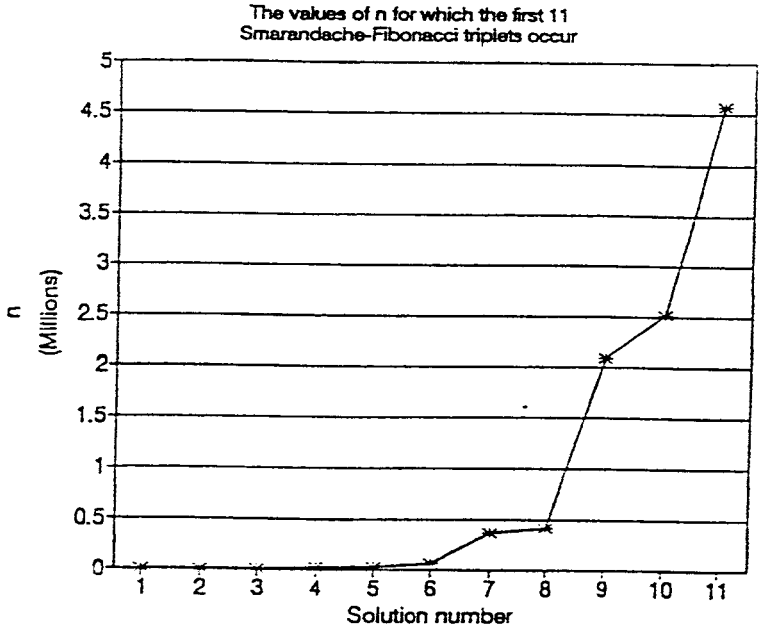

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A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc. is published this year.

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ON CERTAIN INEQUALITIES INVOLVING THE SMARANDACHE FUNCTION

by

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1. The Smarandache function satisfies certain elementary inequalities which have importance in the deduction of properties of this (or related) functions. We quote here the following relations which have appeared in the Smarandache Function Journal:

Let p be a prime number. Then

$$S(p^x) \le S(p^y) \qquad \text{for } x \le y \tag{1}$$

$$\frac{S(p^a)}{p^a} \ge \frac{S(p^{a+1})}{p^{a+1}} \qquad \text{for } a \ge 0 \tag{2}$$

where x, y, a are nonnegative integers;

$$S(p^a) \le S(q^a) \qquad \text{for } p \le q \text{ primes;} \tag{3}$$

$$(p-1)a + 1 \le S(p^a) \le pa\,; \tag{4}$$

If $p > \frac{a}{2}$ and $p \le a-1$ $(a \ge 2)$, then

$$S(p^a) \le p(a-1) \tag{5}$$

For inequalities (3), (4), (5), see [2], and for (1), (2), see [1].

We have also the result ([4]):

$$\text{For composite } n \ne 4\,, \quad \frac{S(n)}{n} \le \frac{2}{3} \tag{6}$$

$$\text{Clearly, } 1 \le S(n) \text{ for } n \ge 1 \text{ and } 1 < S(n) \text{ for } n \ge 2 \tag{7}$$

$$\text{and} \qquad S(n) \le n \tag{8}$$

which follow easily from tfe definition $S(n) = \min \{ k \in N^* : n \text{ divides } k! \}$

2. Inequality (2), written in the form $S(p^{a+1}) \le pS(p^a)$, gives by successive application $S(p^{a+2}) \le pS(p^{a+1}) \le p^2S(p^a)$, ..., that is

$$S(p^{a+c}) \le p^c \cdot S(p^a) \tag{9}$$

where a and c are natural numbers (For c = 0 there is equality, and for a = 0 this follows by (8)).

Relation (9) suggest the following result:

Theorem 1.

For all positive integers m and n holds true the inequality

$$S(mn) \le m \cdot S(n) \tag{10}$$

Proof.

For a general proof, suppose that m and n have a canonical factorization

$$m = p_1^{a_1} \dots p_r^{a_r} \cdot q_1^{b_1} \dots q_s^{b_s}, \quad n = p_1^{c_1} \dots p_r^{c_r} \cdot t_1^{d_1} \dots t_k^{d_k},$$

where $p_i\,(i = \overline{1,r})$, $q_j\,(j = \overline{1,s})$, $t_p\,(p = \overline{1,k})$ are distinct primes and $a_i \ge 0$, $c_j \ge 0$, $b_j \ge 1$, $d_p \ge 1$ are integers.

By a well known result of Smarandache (see [3]) we can write

$$S(m \cdot n) = \max\{S(p_1^{a_1+c_1}), \dots, S(p_r^{a_r+c_r}), S(q_1^{b_1}), \dots, S(q_s^{b_s}), S(t_1^{d_1}), \dots, S(t_k^{d_k})\}$$

$$\le \max\{p_1^{a_1}S(p_1^{c_1}), \dots, p_r^{a_r}S(p_r^{c_r}), S(q_1^{b_1}), \dots, S(q_s^{b_s}), \dots, S(t_k^{d_k})\}$$

by (9). Now, a simple remark and inequality (8) easily give

$$S(m \cdot n) \le p_1^{a_1} \dots p_r^{a_r} q_1^{b_1} \dots q_s^{b_s} \cdot \max\{S(p_1^{c_1}), \dots, S(p_r^{c_r}), S(t_1^{d_1}), \dots, S(t_k^{d_k})\} = mS(n)$$

proving relation (10).

Remark.

For (m,n)=1, inequality (10) appears as

$$\max\{S(m), S(n)\} \le mS(n)$$

This can be proved more generally, for all m and n

Theorem 2.

For all m, n we have:

$$\max\{S(m), S(n)\} \le mS(n) \tag{11}$$

Proof.

The proof is very simple. Indeed, if $S(m) \ge S(n)$, then $S(m) \le mS(n)$ holds, since $S(n) \ge 1$ and $S(m) \le m$, see (7), (8). For $S(m) \le S(n)$ we have $S(n) \le mS(n)$ which is t me by $m \ge 1$. In all cases, relation (11) follows.

This proof has an independent interest. As we shall see, Theorem 2 will follow also from Theorem 1 and the following result:

Theorem 3.

For all m, n we have

$$S(mn) \geq \max \{S(m), S(n)\} \tag{12}$$

Proof.

Inequality (1) implies that $S(p^a) \leq S(p^{a+c})$, $S(p^c) \leq S(p^{a+c})$, so by using the representations of m and n, as in the proof of Theorem 1, by Smarandache's theorem and the above inequalities we get relation (12).

We note that, equality holds in (12) only when all $a_i = 0$ or all $c_i = 0$ $(i = \overline{1,r})$, i.e. when m and n are coprime.

3. As an application of (10), we get:

Corollary 1.

$$\text{a)}\ \frac{S(a)}{a} \leq \frac{S(b)}{b},\ \text{if}\ b \mid a \tag{13}$$

b) If a has a composite divisor $b \neq 4$, then

$$\frac{S(a)}{a} \leq \frac{S(b)}{b} \leq \frac{2}{3} \tag{14}$$

Proof.

Let $a = b \cdot k$. Then $\frac{S(bk)}{bk} \leq \frac{S(b)}{b}$ is equivalent with $S(kb) \leq kS(b)$, which is relation (10) for m=k, n=b.

Relation (14) is a consequence of (13) and (6). We note that (14) offers an improvement of inequality (6).

We now prove:

Corollary 2.

Let m, n, r, s be positive integers. Then:

$$S(mn) + S(rs) \geq \max \{ S(m) + S(r), S(n) + S(s) \} \tag{15}$$

Proof.

We apply the known relation:

$$\max \{ a + c, b + d \} \leq \max \{ a, b \} + \max \{ c, d \} \tag{16}$$

By Theorem 3 we can write $S(mn) \geq \max \{S(m), S(n)\}$ and $S(rs) \geq \max \{S(r), S(s)\}$, so by consideration of (16) with

$$a \equiv S(m),\ b \equiv S(r),\ c \equiv S(n),\ d \equiv S(s)$$

we get the desired result.

Remark.

Since (16) can be generalized to n numbers ($n \geq 2$), and also Theorem 1-3 do hard for the general case (which follow by induction; however these results are based essentially on (10) - (15), we can obtain extensions of these theorems to n numbers.

Corollary 3.

Let a, b composite numbers, $a \neq 4$, $b \neq 4$. Then

$$\frac{S(ab)}{ab} \leq \frac{S(a)+S(b)}{a+b} \leq \frac{2}{3};$$

and

$$S^2(ab) \leq ab[S^2(a) + S^2(b)]$$

where $S^2(a) = (S(a))^2$, etc.

Proof.

By (10) we have $S(a) \geq \frac{S(ab)}{b}$, $S(b) \geq \frac{S(ab)}{a}$, so by addition

$$S(a) + S(b) \geq S(ab)\left(\frac{1}{a} + \frac{1}{b}\right), \text{ giving the first part of (16).}$$

For the second, we have by (6):

$$S(a) \leq \frac{2}{3}a, \; S(b) \leq \frac{2}{3}b, \text{ so } S(a)+S(b) \leq \frac{2}{3}(a+b), \text{ yielding the second part of (16).}$$

For the proof of (17), remark that by $2(n^2 + r^2) \geq (n + r)^2$, the first past of (16), as well as the inequality $2ab \leq (a + b)^2$ we can write successively:

$$S^2(ab) \leq \frac{a^2b^2}{(a+b)^2} \cdot [S(a)+S(b)]^2 \leq \frac{2a^2b^2}{(a+b)^2} \cdot [S^2(a)+S^2(b)] \leq ab[S^2(a)+S^2(b)]$$

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ON SOME CONVERGENT SERIES

by Emil Burton

Notations :

N^* set of integers 1 , 2 , 3 , . . .

$d(n)$ the number of divisors of n .

$S(n)$ the Smarandache function $S : N^* \to N^*$.

$S(n)$ is the smallest integer m with the property that $m!$ is divisible by n

R set of real numbers.

In this article we consider the series $\sum_{k=1}^{\infty} f(S(k))$.

$f : N^* \to R$ is a function which satisfies any conditions.

Proposition 1. Let $f : N \to R$ be a function which satisfies condition :

$$f(t) \leq \frac{c}{t^{\alpha}(d(t!)-d((t-1)!))}$$

for every $t \in N^*$, $\alpha > 1$ constant, $c > 0$ constant.

Then the series $\sum_{k=1}^{\infty} f(S(k))$ is convergent.

Proof: Let us denote by m_t the number of elements of the set

$M_t = \{ k \in N^* ; S(k) = t \} = \{ k \in N^* ; k \mid t! \text{ and } k \nmid (t-1)! \}$.

It follows that $m_t = d(t!) - d((t-1)!)$.

$$\sum_{k=1}^{\infty} f(S(k)) = \sum_{t=1}^{\infty} m_t f(t)$$

We have $m_t \cdot f(t) \leq m_t \cdot \frac{c}{t^{\alpha} m_t} = \frac{c}{t^{\alpha}}$.

It is well - known that $\sum_{t=1}^{\infty} \frac{1}{t^{\alpha}}$ is convergent if $\alpha > 1$.

Therefore $\sum_{k=1}^{\infty} f(S(k)) < \infty$.

It is known that $d(n) < 2\sqrt{n}$ if $n \in N^*$ (2)

and it is obvious that $m_t < d(t!)$ (3)

We can show that

$$\sum_{k=1}^{\infty} (S(k)^p \sqrt{S(k)!})^{-1} < \infty , \quad p > 1 \qquad (4)$$

$$\sum_{k=1}^{\infty} (S(k)!)^{-1} < \infty \qquad (5)$$

$$\sum_{k=1}^{\infty} (S(1)\,!\, S(2)\,!\ldots S(k)\,!\,)^{-1/k} < \infty \qquad (6)$$

$$\sum_{k=2}^{\infty} (S(k)\,\sqrt{S(k)!}\,(\log S(k))^{p}\,)^{-1} < \infty\,,\ p > 1 \qquad (7)$$

Write $f(S(k)) \equiv (S(k)^{p}\cdot\sqrt{S(k)!}\,)^{-1}$, $f(t) \equiv (t^{p}\cdot\sqrt{t!}\,)^{-1} \equiv 2(t^{p}\cdot\sqrt{t!}\,)^{-1} <$
$< 2(t^{p}\cdot d(t!))^{-1} < 2(t^{p}\cdot(d(t!) - d((t-1)!)))^{-1}$.
Now use the proposition 1 to get (4).
The convergence of (5) follows from inequality $t\sqrt{t!} < t!$ if $p \in R$, $p > 1$, $t > t_{*} = [e^{2p+1}]$, $t \in N^{*}$. Here $[e^{2p+1}]$ means the greatest integer $\le e^{2p+1}$.
The details are left to the reader.To show (6) we can use the Carleman's Inequality :Let $(x_n)_{n\in N^{*}}$ be a sequence of positive real numbers and $x_n \neq 0$ for some n. Then

$$\sum_{k=1}^{\infty} (x_1 x_2 \cdots x_k)^{1/k} < e \sum_{k=1}^{\infty} x_k \qquad (8)$$

Write $x_k = (S(k)\,!\,)^{-1}$ and use (8) and (5) to get (6).
It is well-known that

$$\sum_{n=2}^{\infty} (n(\log n)^{p})^{-1} < \infty \quad \text{if and only if } p > 1\,,\ p \in R. \qquad (9)$$

Write $f(t) = (t\sqrt{t!}\,(\log t)^{p})^{-1}$, $t \ge 2$, $t \in N^{*}$. We have

$$\sum_{k=2}^{\infty} (S(k)\sqrt{S(k)!}\,(\log S(k))^{p})^{-1} = \sum_{k=2}^{\infty} m_t f(t)\,.$$

$m_t f(t) < d(t!)\, f(t) < 2\sqrt{t!}\,(t\sqrt{t!}\,(\log t)^{p})^{-1} = 2\,(t\,(\log t)^{p})^{-1}$.
Now use (9) to get (7).
Remark 1. Apply (5) and Cauchy's Condensation Test to see that

$$\sum_{k=0}^{\infty} 2^{k}\,(S(\,2^{k})\,!\,)^{-1} < \infty\,.$$ This implies that $\lim_{k\to\infty} 2^{k}\,(S(\,2^{k})\,!\,)^{-1} = 0$.

A problem :Test the convergence behaviour of the series

$$\sum_{n=1}^{\infty} (S(k)^{p}\sqrt{(S(k)-1)\,!}\,)^{-1}. \qquad (10)$$

Remark 2. This problem is more powerful than (4).
Let p_n denote the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, ...).

It is known that $\sum_{n=1}^{\infty} 1/p_n = \infty$. (11)

We next make use of (11) to obtain the following result :

$$\sum_{n=1}^{\infty} S(n)/n^{2} = \infty. \qquad (12)$$

We have $\sum_{n=1}^{\infty} S(n)/n^2 \geq \sum_{k=1}^{\infty} S(p_k)/p_k^2 = \sum_{k=1}^{\infty} p_k/p_k^2 = \sum_{k=1}^{\infty} 1/p_k$ (13)

Now apply (13) and (11) to get (12).

We can also show that

$$\sum_{n=1}^{\infty} S(n)/n^{1+p} < \infty \text{ if } p > 1 , p \in R . \quad (14)$$

Indeed , $\sum_{n=1}^{\infty} S(n)/n^{1+p} \leq \sum_{n=1}^{\infty} n/n^{1+p} = \sum_{n=1}^{\infty} 1/n^p < \infty$.

If $0 \leq p \leq 2$, we have $S(n)/n^p \geq S(n)/n^2$.

Therefore $\sum_{n=1}^{\infty} S(n)/n^p = \infty$ if $0 \leq p \leq 2$.

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ON SOME DIOPHANTINE EQUATIONS

by

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Let $S(n)$ be defined as the smallest integer such that $(S(n))!$ is divisible by n (Smarandache Function). We shall assume that $S : N^* \to N^*$, $S(1) = 1$. Our purpose is to study a variety of Diophantine equations involving the Smarandache function. We shall determine all solutions of the equations (1), (3) and (8).

(1) $x^{S(x)} = S(x)^x$

(2) $x^{S(y)} = S(y)^x$

(3) $x^{S(x)} + S(x) = S(x)^x + x$

(4) $x^{S(y)} + S(y) = S(y)^x + x$

(5) $S(x)^x + x^2 = x^{S(x)} + S(x)^2$

(6) $S(y)^x + x^2 = x^{S(y)} + S(y)^2$

(7) $S(x)^x + x^3 = x^{S(x)} + S(x)^3$

(8) $S(y)^x + x^3 = x^{S(y)} + S(y)^3$.

For example, let us solve equation (1):

We observe that if $x = S(x)$, then (1) holds.

But $x = S(x)$ if and only if $x \in \{1, 2, 3, 4, 5, 7, \ldots\} = \{x \in N^*; x \text{ -prime}\} \cup \{1, 4\}$.

If $x \geq 6$ is not a prime integer, then $S(x) < x$. We can write $x = S(x) + t$, $t \in N^*$, which implies that $S(x)^{S(x)+t} = (S(x) + t)^{S(x)}$. Thus we have $S(x)^t = \left(1 + \frac{t}{S(t)}\right)^{S(x)}$.

Applying the well - known result $\left(1 + \frac{k}{n}\right)^n < 3^k$, for $n, k \in N^*$, we have $S(x)^t < 3^t$ which implies that $S(x) < 3$ and consequently $x < 3$. This contradicts our choice of x.

Thus, the solutions of (1) are $A_1 = \{x \in N^*; x = \text{prime}\} \cup \{1, 4\}$.

Let us denote by A_k the set of all solutions of the equation (k).

To find A_8 for example, we see that $(S(n), n) \in A_8$ for $n \in N^*$.

Now suppose that $x \neq S(y)$. We can show that (x, y) does not belongs to A_8 as follows: $1 < S(y) < x \Rightarrow S(y) \geq 2$ and $x \geq 3$. On the other hand,

$S(y)^x - x^{S(y)} > S(y)^x - x^x = (S(y) - x)(S(y)^{x-1} + xS(y)^{x-2} + \ldots + x^{x-1}) \geq$

$(S(y) - x)(S(y)^2 + xS(y) + x^2) = S(y)^3 - x^3$.

Thus, $A_8 = \{(x, y); y = n, x = S(n), n \in N^*\}$.

To find A_3, we se that $x = 1$ implies $S(x) = 1$ and (3) holds.

If $S(x) = x$, (3) also holds.

If $x \geq 6$ is not a prime number, then $x > S(x)$.

Write $x = S(x) + t$, $t \in N^* = \{1, 2, 3, \ldots\}$.

Combining this with (3) yields

$S(x)^{S(x)+t} + S(x) + t = (S(x) + t)^{S(x)} + S(x) \Leftrightarrow S(x)^t + t/S(x)^{S(x)} = (1 + t/S(x))^{S(x)} < 3^t$

which implies $S(x) < 3$. This contradicts our choice of x.

Thus $A_3 = \{ x \in N^* ; x = \text{prime} \} \cup \{ 1 , 4 \}$.
Now , we suppose that the reader is able to find $A_2 , A_4 , \ldots , A_7$.
We next determine all positive integers x such that $x = \sum_{k^2 \le x} k^2$

Write $1^2 + 2^2 + \ldots + s^2 = x$ (1)

$s^2 < x$ (2)

$(s+1)^2 \ge x$ (3)

(1) implies $x = s(s+1)(2s+1)/6$. Combining this with (2) and (3) we have $6s^2 < s(s+1)(2s+1)$ and $6(s+1)^2 \ge s(s+1)(2s+1)$. This implies that $s \in \{2 , 3 \}$.
$s = 2 \Rightarrow x = 5$ and $s = 3 \Rightarrow x = 14$.
Thus $x \in \{ 5 , 14 \}$.
In a similar way we can solve the equation $x = \sum_{k^3 < x} k^3$
We find $x \in \{ 9 , 36 , 100 \}$.
We next show that the set $M_p = \{ n \in N^* ; n = \sum_{k^p < n} k^p , p \ge 2 \}$ has at least
$[p/\ln 2] - 2$ elements .
Let $m \in N^*$ such that $m - 1 < p/\ln 2$ (4)
and $p/\ln 2 < m$ (5)
Write (4) and (5) as :
$2 < e^{p/m-1}$ (6)
$e^{p/m} < 2$ (7)
Write $x_k = (1 + 1/k)^k$, $y_k = (1 + 1/k)^{k+1}$.
It is known that $x_s < e < y_t$ for every $s , t \in N^*$.
Combining this with (6) and (7) we have
$x_s^{p/m} < e^{p/m} < 2 < e^{p/m-1} < y_t^{p/m-1}$ for every $s , t \in N^*$.
We have $2 < y_t^{p/m-1} = ((t+1)/t)^{(t+1)p/m-1} \le ((t+1)/t)^p$ if $(t+1)/(m-1) \le 1$.
So , if $t \le m - 2$ we have $2 < ((t+1)/t)^p \Leftrightarrow 2t^p < (t+1)^p \Leftrightarrow (t+1)^p - t^p > t^p$ (8).
Let $A_p(s)$ denote the sum $1^p + 2^p + \ldots + s^p$.
Proposition 1. $(t+1)^p > A_p(t)$ for every $t \le m-2 , t \in N^*$.
Proof. Suppose that $A_p(t) \ge (t+1)^p \Leftrightarrow A_p(t-1) > (t+1)^p - t^p > t^p \Leftrightarrow$
$A_p(t-2) > t^p - (t-1)^p > (t-1)^p \Leftrightarrow \ldots \Leftrightarrow A_p(1) > 2^p$ which is not true.
It is obvious that $A_p(t) > t^p$ if $t \in N^*$, $2 \le t \le m-2$ which implies $A_p(t) \in M_p$ for every $t \in N^*$ and $2 \le t \le m-2$.
Therefore card $M_p > m-3 = (m-1) - 2 = [p/\ln 2] - 2$.

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ABOUT THE SMARANDACHE COMPLEMENTARY PRIME FUNCTION

by

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Let $c : N \to N$ be the function defined by the condition that $n + c(n) = p_i$, where p_i is the smallest prime number, $p_i \geq n$.

Example

$c(0) = 2,\ c(1) = 1,\ c(2) = 0,\ c(3) = 0,\ c(4) = 1,\ c(5) = 0,\ c(6) = 1,$ $c(7) = 0$ and so on.

1) If p_k and p_{k-1} are two consecutive primes and $p_k < n \leq p_{k-1}$, then :

$c(n) \in \{ p_{k-1} - p_k - 1, p_{k-1} - p_k - 2, \ldots, 1, 0 \}$, because :

$c(p_k + 1) = p_{k-1} - p_k - 1$ and so on, $c(p_{k-1}) = 0$.

2) $c(p) = c(p - 1) - 1 = 0$ for every p prime, because $c(p) = 0$ and $c(p - 1) = 1$.

We also can observe that $c(n) \neq c(n + 1)$ for every $n \in N$.

1. Property

The equation $c(n) = n$, $n > 1$ has no solutions.

Proof

If n is a prime it results $c(n) = 0 < n$.

It is wellknown that between n and $2n$, $n > 1$ there exists at least a prime number. Let p_k be the smallest prime of them. Then if n is a composite number we have :

$c(n) = p_k - n < 2n - n = n$, therefore $c(n) < n$.

It results that for every $n \neq p$, where p is a prime, we have $\frac{1}{n} \leq \frac{c(n)}{n} < 1$, therefore $\sum_{\substack{n \neq p \\ p \text{ prime}}} \frac{c(n)}{n}$ diverges. Because for the primes $c(p) / p = 0$ we can say that $\sum_{n \geq 1} \frac{c(n)}{n}$ diverges.

2. Property

If n is a composite number, then $c(n) = c(n-1) + 1$.

Proof

Obviously.

It results that for n and $(n-1)$ composite numbers we have $\frac{c(n)}{c(n-1)} > 1$. Now, if $p_k < n < p_{k+1}$ where p_k and p_{k+1} are consecutive primes, then we have :

$c(n)\, c(n-1) \ldots c(p_{k+1} - 1) = (p_{k+1} - n)!$

and if $n \leq p_k < p_{k+1}$ then $c(n)\, c(n-1) \ldots c(p_{k+1} - 1) = 0$.

Of course, every $\prod_{n=k}^{r} c(n) = 0$ if there exists a prime number p, $k \leq p \leq r$.

If $n = p_k$ is any prime number, then $c(n) = 0$ and because $c(n-1) = p_{k+1} - n - 1$ it results that $|c(n) - c(n+1)| = 1$ if and only if n and $(n+2)$ are primes (friend prime numbers)

3. Property

For every k - th prime number p_k we have :

$c(p_k - 1) < (\log p_k)^2 - 1$.

Proof

Because $c(p_k - 1) = p_{k+1} - p_k - 1$ we have $p_{k+1} - p_k = c(p_k - 1) + 1$.

But, on the other hand we have $p_{k+1} - p_k < (\log p_k)^2$, then the assertion follows.

4. Property

$c(c(n)) < c(n)$ and $c^m(n) < c(n) < n$, for every $n > 1$ and $m \geq 2$.

Proof

If we denote $c(n) = r$ then we have :

$c(c(n)) = c(r) < r = c(n)$.

Then we suppose that the assertion is true for m : $c^m(n) < c(n) < n$, and we prove it

for (m − 1), too :

$c^{m-1}(n) = c(c^m(n)) < c^m(n) < c(n) < n.$

5. Property

For every prime p we have $(c(p-1))^n \leq c((p-1)^n)$.

Proof

$c(p-1) = 1 \Rightarrow (c(p-1))^n = 1$ while $(p-1)^n$ is a composite number, therefore $c((p-1)^n) \geq 1$.

6. Property

The following kind of Fibonacci equation :

$$c(n) - c(n-1) = c(n+2) \qquad (1)$$

has solutions.

Proof

If n and (n + 1) are both composite numbers, then $c(n) > c(n+1) \geq 1$. If (n + 2) is a prime, then c (n + 2) = 0 and we have no solutions in this case. If (n + 2) is also a composite number, then :

$c(n) > c(n-1) > c(n+2) \geq 1$, therefore $c(n) + c(n+1) > c(n+2)$ and we have no solutions also in this case.

Therefore n and (n + 1) are not both composite numbers in the equality (1).

If n is a prime, then (n + 1) is a composite number and we must have :

$0 - c(n-1) = c(n+2)$, wich is not possible (see (2)).

We have only the case when (n + 1) is a prime; in this case we must have :

$1 - 0 = c(n+2)$ but this implies that (n + 3) is a prime number, so the only solutions are when (n + 1) and (n − 2) are friend prime numbers.

7. Property

The following equation:

$$\frac{c(n) + c(n+2)}{2} = c(n+1) \qquad (2)$$

has an infinite number of solutions.

Proof

Let p_k and p_{k-1} be two consecutive prime numbers, but not friend prime numbers.

Then, for every integer i between $p_k + 1$ and $p_{k-1} - 1$ we have:

$$\frac{c(i-1)+c(i-1)}{2} = \frac{(p_{k+1}-i+1)+(p_{k+1}-i-1)}{2} = p_{k+1}-i = c(i).$$

So, for the equation (2) all positive integer n between $p_k + 1$ and $p_{k-1} - 1$ is a solution.

If n is prime, the equation becomes $\frac{c(n+2)}{2} = c(n+1)$.

But $(n+1)$ is a composite number, therefore $c(n+1) \neq 0 \Rightarrow c(n+2)$ must be composite number. Because in this case $c(n+1) = c(n+2) + 1$ and the equation has the form $\frac{c(n+2)}{2} = c(n+2) + 1$, so we have no solutions.

If $(n+1)$ is prime, then we must have $\frac{c(n)+c(n+2)}{2} = 0$, where n and $(n+2)$ are composite numbers. So we have no solutions in this case, because $c(n) \geq 1$ and $c(n+2) \geq 1$.

If $(n+2)$ is a prime, the equation has the form $\frac{c(n)}{2} = c(n+1)$, where $(n+1)$ is a composite number, therefore $c(n+1) \neq 0$. From (2) it rezults that $c(n) \neq 0$, so n is also a composite number. This case is the same with the first considered case.

Therefore the only solutions are for $\overline{p_k, p_{k+1}-2}$, where p_k, p_{k-1} are consecutive primes, but not friend consecutive primes.

8. Property

The greatest common divisor of n and $c(x)$ is 1 :

$(x, c(x)) = 1$, for every composite number x.

Proof

Taking into account of the definition of the function c, we have $x + c(x) = p$, where p is a prime number.

If there exists $d \neq 1$ so that d / x and $d / c(x)$, then it implies that d / p. But p is a prime number, therefore $d = p$.

This is not possibile because $c(x) < p$.

If p is a prime number, then $(p, c(p)) = (p, 0) = p$.

9. Property

The equation $[x, y] = [c (x), c (y)]$, where $[x, y]$ is the least common multiple of x and y has no solutions for $x, y > 1$.

Proof

Let us suppose that $x = dk_1$ and $y = dk_2$, where $d = (x, y)$. Then we must have :

$[x, y] = dk_1 k_2 = [c (x), c (y)]$.

But $(x, c (x)) = (dk_1 , c (x)) = 1$, therefore dk_1 is given in the least common multiple $[c (x), c (y)]$ by $c (y)$.

But $(y, c (y)) = (dk_1 , c (y)) = 1 \Rightarrow d = 1 \Rightarrow (x, y) = 1 \Rightarrow$

$\Rightarrow [x, y] = xy > c (x) c (y) \geq [c (x), c (y)]$, therefore the above equation has no solutions, for $x, y > 1$.

For $x = 1 = y$ we have $[x, y] = [c (x), c (y)] = 1$.

10. Property

The equation :

$$(x, y) = (c (x), c (y)) \qquad (3)$$

has an infinite number of solutions.

Proof

If $x = 1$ and $y = p - 1$ then $(x, y) = 1$ and $(c (x), c (y)) = (1, 1) = 1$, for an arbitrary prime p.

Easily we observe that every pair $(n, n - 1)$ of numbers is a solutions for the equation (3), if n is not a prime.

11. Property

The equation :

$$c (x) - x = c (y) - y \qquad (4)$$

has an infinite number of solutions.

Proof

From the definition of the function c it results that for every x and y satisfying

$p_k < x \le y \le p_{k-1}$ we have $c(x) + x = c(y) + y = p_{k-1}$. Therefore we have $(p_{k-1} - p_k)^2$ couples (x, y) as different solutions. Then, until the n-th prime p_n, we have $\sum_{k=1}^{n-1} (p_{k-1} - p_k)^2$ different solutions.

Remark

It seems that the equation $c(x) + y = c(y) + x$ has no solutions $x \neq y$, but it is not true.

Indeed, let p_k and p_{k-1} be consecutive primes such that $p_{k-1} - p_k = 6$ (is possibile : for example $29 - 23 = 6$, $37 - 31 = 6$, $53 - 47 = 6$ and so on) and $p_k - 2$ is not a prime.

Then $c(p_k - 2) = 2$, $c(p_k - 1) = 1$, $c(p_k) = 0$, $c(p_k + 1) = 5$, $c(p_k + 2) = 4$, $c(p_k + 3) = 3$ and we have :

1. $c(p_k + 1) - c(p_k - 2) = 5 - 2 = 3 = (p_k + 1) - (p_k - 2)$
2. $c(p_k + 2) - c(p_k - 1) = 3 = (p_k + 2) - (p_k - 1)$
3. $c(p_k + 3) - c(p_k) = 3 = (p_k + 3) - p_k$, thus

$c(x) - c(y) = x - y$ ($\Leftrightarrow c(x) + y = c(y) + x$) has the above solutions if $p_k - p_{k-1} > 3$

If $p_k - p_{k-1} = 2$ we have only the two last solutions.

In the general case, when $p_{k-1} - p_k = 2h$, $h \in N^*$, let $x = p_k - u$ and $y = p_k + v$, $u, v \in N$ be the solutions of the above equation.

Then $c(x) = c(p_k - u) = u$ and $c(y) = c(p_k + v) = 2h - v$.

The equation becomes:

$u + (p_k + v) = (2h - v) + (p_k - u)$, thus $u + v = h$.

Therefore, the solutions are $x = p_k - u$ and $y = p_k + h - u$, for every $u = \overline{0, h}$ if $p_k - p_{k-1} > h$ and $x = p_k - u$, $y = p_k + h - u$, for every $u = \overline{0, l}$ if $p_k - p_{k-1} = l + 1 \le h$.

Remark

$c(p_k + 1)$ is an odd number, because if p_k and p_{k-1} are consecutive primes, $p_k > 2$, then p_k and p_{k-1} are, of course, odd numbers; then $p_{k-1} - p_k - 1 = c(p_k + 1)$ are **always** odd.

12. Property

The sumatory function of c, $F_c(n) \overset{def}{=} \sum_{\substack{d \in N \\ d/n}} c(d)$ has the properties :

a) $F_c(2p) = 1 + c(2p)$

b) $F_c(pq) = 1 + c(pq)$, where p and q are prime numbers.

Proof

a) $F_c(2p) = c(1) + c(2) + c(p) + c(2p) = 1 + c(2p)$.

b) $F_c(pq) = c(1) + c(p) + c(q) + c(pq) = 1 + c(pq)$.

Remark

The function c is not multiplicative : $0 = c(2) \cdot c(p) < c(2p)$.

13. Property

$$c^k(p) = \begin{cases} 0 & \text{for } k \text{ odd number} \\ 2 & \text{for } k \text{ even number, } k \geq 1 \end{cases}$$

Proof

We have :

$c^1(p) = 0$;

$c^2(p) = c(c(p)) = c(0) = 2$;

$c^3(p) = c(2) = 0$;

$c^4(p) = c(0) = 2$.

Using the complete mathematical induction, the property holds.

Consequences

1) We have $\dfrac{c^k(p) + c^{k+1}(p)}{2} = 1$ for every $k \geq 1$ and p prime number.

2) $\sum_{k=1}^{r} c^k(p) = \left[\frac{r}{2}\right] \cdot 2$, where $[x]$ is the integer part of x, and

$\sum_{\substack{k=2 \\ k \text{ even}}}^{r} \dfrac{1}{c^k(p)} = \left[\frac{r}{2}\right] \cdot \frac{1}{2}$, thus $\sum_{k \geq 1} c^k(p)$ and $\sum_{\substack{k \geq 2 \\ k \text{ even}}} \dfrac{1}{c^k(p)}$ are divergent series.

Remark

$c^k(p - 1) = c^{k-1}(c(p - 1)) = c^{k-1}(1) = 1$, for every prime $p > 3$ and $k \in N^*$, therefore $c^{k_1}(p_1 - 1) = c^{k_2}(p_2 - 1)$ for every primes $p_1, p_2 > 3$ and $k_1, k_2 \in N^*$.

14. Property

The equation :

$$c(x) + c(y) + c(z) = c(x)\,c(y)\,c(z) \qquad (5)$$

has an infinite number of solutions.

Proof

The only non-negative solutions for the diofantine equation $a + b + c = abc$ are $a = 1$, $b = 2$ and $c = 3$ and all circular permutations of $\{ 1, 2, 3 \}$.

Then :

$c (x) = 1 \Rightarrow x = p_k - 1$, p_k prime number, $p_k > 3$

$c (y) = 2 \Rightarrow y = p_k - 2$, where p_{r-1} and p_r are consecutive prime numbers such that $p_r - p_{r-1} \geq 3$

$c (z) = 3 \Rightarrow z = p_t - 3$, where p_{t-1} and p_t are consecutive prime numbers such that $p_t - p_{t-1} \geq 4$

and all circular permutations of the above values of x, y and z.

Of course, the equation $c (x) = c (y)$ has an infinite number of solutions.

Remark

We can consider $c^{\leftarrow} (y)$, for every $y \in N^*$, defined as $c^{\leftarrow} (y) = \{ x \in N \mid c (x) = y \}$. For example $c^{\leftarrow} (0)$ is the set of all primes, and $c^{\leftarrow} (1)$ is the set $\{ 1, p_{k-1} \}_{\substack{p_k \text{ prime} \\ p_k > 3}}$ and so on.

A study of these sets may be interesting.

Remark

If we have the equation :

$$c^k (x) = c (y), k \geq 2 \qquad (6)$$

then, using property 13, we have two cases.

If x is prime and k is odd, then $c^k (x) = 0$ and (5) implies that y is prime.

In the case when x is prime and k is even it results $c^k (x) = 2 = c (y)$, which implies that y is a prime, such that $y - 2$ is not prime.

If $x = p$, $y = q$, p and q primes, $p,q > 3$, then $(p - 1, q - 1)$ are also solutions, because $c^k (p - 1) = 1 = c (q - 1)$, so the above equation has an infinite number of couples as solutions.

Also a study of $(c^k (x))^{\leftarrow}$ seems to be interesting.

Remark

The equation :

$$c(n) + c(n-1) + c(n+2) = c(n-1) \qquad (7)$$

has solutions when $c(n-1) = 3$, $c(n) = 2$, $c(n+1) = 1$, $c(n+2) = 0$, so the solutions are $n = p - 2$ for every p prime number such that between $p - 4$ and p there is not another prime.

The equation :

$$c(n-2) + c(n-1) + c(n+1) + c(n+2) = 4c(n) \qquad (8)$$

has as solutions $n = p - 3$, where p is a prime such that between $p - 6$ and p there is not another prime, because $4c(n) = 12$ and $c(n-2) + c(n-1) + c(n+1) + c(n+2) = 12$.

For example $n = 29 - 3 = 26$ is a solution of the equation (7).

The equation :

$$c(n) + c(n-1) + c(n-2) + c(n-3) + c(n-4) = 2c(n-5) \qquad (9)$$

(see property 7) has as solution $n = p - 5$, where p is a prime, such that between $p - 6$ and p there is not another prime. Indeed we have $0 + 1 + 2 + 3 + 4 = 2 \cdot 5$.

Thus, using the properties of the function c we can decide if an equation, which has a similar form with the above equations, has or has not solutions.

But a difficult problem is : " For any even number a, can we find consecutive primes such that $p_{k-1} - p_k = a$? "

The answer is useful to find the solutions of the above kind of equations, but is also important to give the answer in order to solve another open problem :

" Can we get, as large as we want, but finite decreasing sequence k, k - 1, ..., 2, 1, 0 (**odd k**), included in the sequence of the values of c?"

If someone gives an answer to this problem, then it is easy to give the answer (it will be the same) at the similar following problem :

" Can we get, as large as we want, but finite decreasing sequence k, k - 1, ..., 2, 1, 0 (**even k**), included in the sequence of the values of c?"

We suppose the answer is negative.

In the same order of ideea, it is interesting to find $\max_n \frac{c(n)}{n}$.

It is wellknown (see [4], page 147) that $p_{n+1} - p_n < (\ln p_n)^2$, where p_n and p_{n+1} are two consecutive primes.

Moreover, $\frac{c(n)}{n}$, $p_k < n \le p_{k+1}$ reaches its maximum value for $n = p_k - 1$, where p_k is a prime.

So, in this case :

$$\frac{c(n)}{n} = \frac{p_{k+1} - p_k - 1}{p_k - 1} < \frac{(\ln p_k)^2 - 1}{p_k - 1} \overset{k\to\infty}{\to} 0$$

Using this result, we can find the maximum value of $\frac{c(n)}{n}$

For $p > 100$ we have $\frac{(\ln p)^2 - 1}{p+1} < \frac{(\ln 100)^2 - 1}{101} < \frac{1}{4}$.

Using the computer, by a straight forward computation, it is easy to prove that $\max_{2\le n\le 100} \frac{c(n)}{n} = \frac{3}{8}$, wich is reached for $n = 8$.

Because $\frac{c(n)}{n} < \frac{1}{4}$ for every $n>100$ it results that $\max_{n\ge 2} \frac{c(n)}{n} = \frac{3}{8}$ reached for $n = 8$.

Remark

There exists an infinite number of finite sequences $\{ c(k_1), c(k_1 + 1), ..., c(k_2) \}$ such that $\sum_{k=k_1}^{k_2} c(k)$ is a three-cornered number for $k_1, k_2 \in N^*$ (the n-th three-cornered number is $T_n \overset{def}{=} \frac{n(n+1)}{2}$, $n \in N^*$).

For example, in the case $k_1 = p_k$ and $k_2 = p_{k+1}$, two consecutive primes, we have the finite sequence $\{ c(p_k), c(p_k + 1), ..., c(p_{k+1} - 1), c(p_{k+1}) \}$ and

$$\sum_{k=p_k}^{p_{k+1}} c(k) = 0 + (p_{k+1} - p_k - 1) + .. + 2 + 1 + 0 = \frac{(p_{k+1} - p_k - 1)(p_{k+1} - p_k)}{2} = T_{p_{k+1} - p_k - 1}$$

Of course, we can define the function $c' : N \setminus \{ 0, 1 \} \to N$, $c'(n) = n - k$, where k is the smallest natural number such that $n - k$ is a prime number, but we shall give some properties of this function in another paper.

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THE FUNCTIONS $\theta_s(x)$ AND $\tilde{\theta}_s(x)$

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In this paper we define the function $\theta_s: N \setminus \{0,1\} \to N$ and $\tilde{\theta}_s: N \setminus \{0,1,2\} \to N$ as follows :

$$\theta_s(x) = \sum_{\substack{p/x \\ 0<p\le x \\ p-\text{prime}}} S(p^x); \quad \tilde{\theta}_s(x) = \sum_{\substack{p\nmid x \\ 0<p\le x \\ p-\text{prime}}} S(p^x),$$

where $S(p^x)$is the Smarandache function defined in [3] (S(n) is the smollest integer m such that m! is divisible by n).

For the begining we give some properties of the θ function. Let us observe that, from the definition of θ_s it results:

$\theta_s(2) = S(2^2) = 4$,
$\theta_s(3) = S(3^3) = 6$,
$\theta_s(4) = S(2^4) = 6$,
$\theta_s(5) = S(5^5) = 25$,
$\theta_s(6) = S(2^6) + S(3^6) = 7 + 15 = 22$,
$\theta_s(7) = S(7^7) = 49$,
$\theta_s(8) = S(2^8) = 10$,
$\theta_s(9) = S(3^9) = 21$,
$\theta_s(10) = S(2^{10}) + S(5^{10}) = 12 + 45 = 57$,
$\theta_s(11) = S(11^{11}) = 121$,
$\theta_s(12) = S(2^{17}) + S(3^{12}) = 43$.

We note also that if p-prime than $\theta_s(p^p) = p^2$.

Proposition 1. *The series* $\sum_{x\ge 2} (\theta_s(x))^{-1}$ *is convergent.*

Proof. $$\sum_{x\ge 2} (\theta_s(x))^{-1} = \frac{1}{S(2^2)} + \frac{1}{S(3^3)} + \frac{1}{S(5^5)} + \frac{1}{S(2^6)+S(3^6)} +$$

$$+ \frac{1}{S(7^7)} + \frac{1}{S(2^8)} + \frac{1}{S(3^9)} + \frac{1}{S(2^{10})+S(5^{10})} + \frac{1}{S(11^{11})} + \cdots$$

$$\le \sum_{\substack{i=2 \\ p_i/x}}^{\infty} \left(\frac{1}{p_i^2} + \frac{1}{(p_{V(x)} - 1)V(x)} \right) = \sum_{i=2}^{\infty} \frac{1}{p_i^2} + \sum_{p_{V(x)}/x} \frac{1}{(p_{V(x)} - 1)V(x)},$$

where V(x) denote the number of the primes less or equal with x and divide by x.

Of course the series $\sum_{i=2}^{\infty}\frac{1}{p_i^2}$ and $\sum_{p_{V(x)}/x}\frac{1}{(p_{V(x)}-1)V(x)}$ are convergent, so the propositionis proved.

Proposition 2. *Let the sequence* $T(n)=1-\lg\theta_s(n)+\sum_{i=2}^{n}\frac{1}{\theta_s(i)}$. *Then* $\lim_{n\to\infty}T(n)=-\infty$.

The proof is imediate because the series $\sum_{n=2}^{\infty}\frac{1}{\theta_s(n)}$ is convergent according by the proposition 1.

Proposition 3. *The equation* $\theta_s(x)=\theta_s(x+1)$ (0) *has no solution if x is a prime.*

Proof. If x is a prime number the equation become

$x^2=\theta_s(x+1)$, where

$$\theta_s(x+1)=S(p_{i_1}^{x+1})+S(p_{i_2}^{x+1})+\cdots+S(p_{i_{V(x+1)}}^{x+1}).$$

Using the inequality

$$(p-1)\alpha<S(p^{\alpha})\le p\alpha \qquad (1)$$

given in [4], we have

$$\theta_s(x+1)\le(x+1)(p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}})$$

Let us presume that the equation (0) has solution. We have the following relation:

$$x^2\le(x+1)(p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}}) \qquad (2)$$

and we prove that

$$p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}}\le x-1 \qquad (3)$$

for $x\ge 9$.

Let $\eta=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots\cdot p_r^{\alpha_r}$, $p_i\ne p_j$, $i\ne j$, the decomposition of n into primes. We define the function $f(n)=1+\alpha_1p_1+\cdots+\alpha_rp_r$ and we show that $f(n)\le n-2$ for $n\ge 9$. If $1\le n<9$ the precedent inequality is verified by calculus). For $n\ge 9$, we prove the inequality by induction:

$f(9) = 7,\ f(10) = 8,\ f(12) = 8 < 10$, true.

Now let us suppose that $f(n) \le n-2,\ \forall\, n \ge 12$, and we show that $f(n+1) \le n-1$.

In this case we have three different situations:

I) $n+1 = h = k_1 \cdot k_2$, where k_1, k_2 are composed members. Using the true relation, $f(h) = f(k_1 \cdot k_2) = f(k_1) + f(k_2) - 1$, we have

$$f(h) = f(n+1) = f(k_1) + f(k_2) - 1 \le k_1 - 2 + k_2 - 2 - 1 = k_1 \cdot k_2 - 8 - 5 \le$$
$$\le h - 2 = n + 1 - 2 = n - 1 \Rightarrow f(n+1) \le n - 1.$$

II) $n+1 = h = k_1 \cdot k_2$, where, k_1 – prime, k_2 – compounded,

$$f(h) = f(k_1) + f(k_2) - 1 \le k_1 + 1 + k_2 - 2 - 1 \le k_1 \cdot k_2 - 2 = n - 1.$$

III) $n+1 = h = k_1 \cdot k_2$, where k_1, k_2 – prime,

$$f(h) = 1 + k_1 + k_2 = k_1 k_2 + 2 - (k_1 - 1)(k_2 - 1) \le h + 2 - 4 = h - 2 = n - 1.$$

Conclusion: $f(n) \le n-2,\ \forall\, n \ge 9$.

Then $f(n) \le n+2 \Rightarrow 1 + \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 2 \Rightarrow$

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 3.$$

We obtain

$$p_1 + p_2 + \cdots + p_r \le \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 3 \le n - 2$$

Using (3) in (2) we have

$$x^2 \le (x+1)(x-1) \Rightarrow x^2 < x^2 - 1, \text{ imposible.}$$

Proposition 4. *The equation* $\theta_s(x) = \theta_s(x+1)$ *has no solution for* $(x+1)$ *- prime.*

Proof. We have $\theta_s(x+1) = (x+1)^2$.

We suppose that the equation has solution and with the inequalityes (1) is must that

$$(x+1)^2 \le x(p_{i_1}^x + p_{i_2}^x + \cdots + p_{i_{\nu(x)}}^x) \le x^2, \text{ imposible.}$$

We give some particular value for $\tilde{\theta}_s(x) = \sum_{\substack{p^x \le x \\ p-\text{prime}}} S(p^x)$;

$\tilde{\theta}_s(3) = S(2^3) = 4$

$\tilde{\theta}_s(4) = S(3^4) = 9$

$\tilde{\theta}_s(5) = S(2^5) + S(3^5) = 20$

$\tilde{\theta}_s(6) = S(5^6) = 30$

$\tilde{\theta}_s(7) = S(2^7) + S(3^7) + S(5^7) =$
$= 8+18+36=62$

$\tilde{\theta}_s(8) = S(3^8) + S(5^8) + S(7^8) = 18+35+49 = 102$

$\tilde{\theta}_s(9) = S(2^9) + S(5^9) + S(7^9) = 12+40+56 = 108$

$\tilde{\theta}_s(10) = S(3^{10}) + S(7^{10}) = 24+63 = 87$

$\tilde{\theta}_s(11) = S(2^{11}) + S(3^{11}) + S(5^{11}) + S(7^{11}) = 16+27+$
$+50+73=163$

$\tilde{\theta}_s(12) = S(2^{12}) + S(7^{12}) + S(11^{12}) = 50+77+$
$+121=248$

$\tilde{\theta}_s(13) = S(2^{13}) + S(3^{13}) + S(5^{13}) + S(7^{13}) + S(11^{13}) =$
$= 16+27+60+84+132=319.$

Proposition 5. *The series* $\sum_{x\geq 3}(\tilde{\theta}_s(x))^{-1}$ *is convergent.*

Proof.

$$\sum_{x\geq 3}(\tilde{\theta}_s(x))^{-1} = \frac{1}{S(2^3)} + \frac{1}{S(3^4)} + \frac{1}{S(2^5)+S(3^5)} + \frac{1}{S(2^8)+S(5^8)+S(7^8)} + \frac{1}{S(2^9)+S(5^9)+S(7^9)}$$

$$+\cdots \leq \frac{1}{S(2^3)} + \frac{1}{S(3^4)} + \frac{1}{S(5^6)} + \frac{1}{S(2^7)} + \cdots \leq \sum_{\substack{x\geq 3 \\ p_i^x \mid x \\ p_i-\text{prime}}} \frac{1}{S(p_i^x)} \leq \sum_{\substack{x\geq 3 \\ p_i^2 \mid x \\ p_i-\text{prime}}} \frac{1}{p_i^2} \leq \sum_{x\geq 3}\frac{1}{x^2}.$$

Because the series $\sum_{x\geq 3}\frac{1}{x^2}$ is convergent, we have that our series is convergent.

Proposition 6. *If* $T(n) = 1 - \lg \tilde{\theta}_s(n) + \sum_{i=3}^{n}\frac{1}{\tilde{\theta}_s(i)}$ *then* $\lim_{n\to\infty} T(n) = -\infty$.

Proposition 7. *The equation* $\tilde{\theta}_s(x) = \tilde{\theta}_s(x+1)$ *has no solution if* x+1=p-*prime.*

Proof. If x+1 is prime be wouldn't divide with any of prime numbers then him

$$\bar{\theta}_s(x+1) = \sum_{\substack{p^{x+1} \\ 0<p\le x+1}} S(p^{x+1}) = S(p_{i_1}^{x+1}) + S(p_{i_2}^{x+1}) + \cdots + S(p_{i_{\pi(x)}}^{x+1}).$$

The number x is divisible with at least two prime numbers then him. In the case $\bar{\theta}_s(x) = \sum_{\substack{p^x x \\ 0<p\le x}} S(p^x)$ will have at least two terms $S(p_{i_k}^x)$ less then they are in $\bar{\theta}_s(x+1)$.

Moreover $S(p_i^x) \le S(p_i^{x+1})$ and it results that $\bar{\theta}_s(x) < \bar{\theta}_s(x+1)$.

Proposition 8. *The equation* $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ *has no solution if* x=p-prime, $x \ge 9$.

Proof. using the function $F_s(x) = \sum_{\substack{0<p\le x \\ p-\text{prime}}} S(p^x)$ defined in [2] we have

$$F_s(x) = \theta_s(x) + \bar{\theta}_s(x)$$

$$F_s(x+1) = \theta_s(x+1) + \bar{\theta}_s(x+1).$$

If our equation have solution $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ then

$$F_s(x) - F_s(x+1) = \theta_s(x) - \theta_s(x+1)$$

or

$$F_s(x) - F_s(x+1) = x^2 - \theta_s(x+1).$$

Is known [2] that $F_s(x) - F_s(x+1) < 0$. We have $x^2 - \theta_s(x+1) < 0 \Rightarrow x^2 < \theta_s(x+1)$. Using (3) we have

$\theta_s(x+1) \le (x+1)(x-1) = x^2 - 1$, therefore $x^2 < x^2 - 1$, imposible.

For x<9 is verified by calculus that the equation $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ has not solution.

Proposed problem

1. $\theta_s(x) = \theta_s(x+1)$, x, x+1 are composed numbers.
2. $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$, x, x+1, are composed numbers.

Calculate

3. $\lim_{n\to\infty} \frac{\theta_s(n)}{n^\alpha}$, $\alpha \in \mathbf{R}$.

4. $\lim_{n\to\infty} \frac{\bar{\theta}_s(n)}{n^\alpha}$, $\alpha \in \mathbf{R}$.

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The function $\Pi_S(x)$

by

Vasile Seleacu and Stefan Zanfir

In this paper are studied some properties of the numerical function $\Pi_S : N^* \to N$, $\Pi_S(x) = \{ m \in (0, x] / S (m) = \text{prime number} \}$, where $S (m)$ is the Smarandache function, defined in [1].

Numerical example :

$\Pi_S (1) = 0$, $\Pi_S (2) = 1$, $\Pi_S (3) = 2$, $\Pi_S (4) = 2$, $\Pi_S (5) = 3$, $\Pi_S (6) = 4$, $\Pi_S (7) = 5$, $\Pi_S (8) = 5$, $\Pi_S (9) = 5$, $\Pi_S (10) = 6$, $\Pi_S (11) = 7$, $\Pi_S (12) = 7$, $\Pi_S (13) = 8$, $\Pi_S (14) = 9$, $\Pi_S (15) = 10$, $\Pi_S (16) = 10$, $\Pi_S (17) = 11$, $\Pi_S (18) = 11$, $\Pi_S (19) = 12$, $\Pi_S (20) = 13$.

Proposion 1.

According to the definition we have :

a) $\Pi_S (x) \leq \Pi_S (x + 1)$,

b) $\Pi_S (x) = \Pi_S (x - 1) + 1$, if x is a prime,

c) $\Pi_S (x) \leq \varphi (x)$, if x is a prime,

where $\varphi (x)$ is the Euler's totient function.

Proposition 2.

The equation $\Pi_S (x) = \left[\frac{x}{2}\right]$, in the hypothesis $x \neq 1$ and $\Pi_S (x + 1) = \Pi_S (x)$ has no solution in the following situation :

a) x is a prime,

b) x is a composite number, odd

c) $x + 1$ is the square of a positiv integer and x is odd.

Proof.

Using the reduction ad absurdum method we suppose that the equation $\Pi_S(x) = \left[\frac{x}{2}\right]$ has solution. Then $\Pi_S(x+1) = \left[\frac{x+1}{2}\right]$. Using the hypothesis we have :

$\left[\frac{x}{2}\right] = \left[\frac{x+1}{2}\right]$, false.

Because $x + 1$ is a perfect square we deduce that x is a composite number and because x is an uneven we obtain (b).

Proposition 3.

$\forall\, a \geq 2$ and $k \geq 2$ $S(a^k)$ is not a prime.

Proof.

If we suppose that $S(a^k) = p$ is a prime, then $p! = a^k p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} p$ and $(a^k, p) = 1$. We deduce that $a^k / (p-1)! \Rightarrow$

$S(a^k) \leq p - 1 < p$, false.

Proposition 4.

$\forall\, x \in N^*$, we have :

$$\left[\frac{x}{2}\right] \leq \Pi_S(x) \leq x - [\sqrt{x}]$$

Proof.

We used the mathematical induction. In the particular case $x \in \{1, 2, 3, 4\}$ our inequality is verified by direct calculus.

We suppose that the inequality is verified for $x \in N^*$ and we proved it for $x + 1$.

We have the following cases :

1) $x + 1$ the prime number, with the subcases :

a) x is not a square of some integer. Then $\Pi_S(x+1) = \Pi_S(x) + 1$.

We suppose that $\Pi_S(x) \leq x - [\sqrt{x}]$

Let prove that $\Pi_S(x+1) \leq x + 1 - [\sqrt{x+1}]$.

It results that $\Pi_S(x+1) \leq x + 1 - [\sqrt{x+1}] \Leftrightarrow \Pi_S(x) \leq x - [\sqrt{x+1}]$. It's enough to prove that $x - [\sqrt{x}] \leq x - [\sqrt{x+1}]$. This relation is true because from our hypothesis it results that $[\sqrt{x}] = [\sqrt{x+1}]$.

For the left side of the inequality we have $\Pi_S(x) \geq \left[\frac{x}{2}\right]$, true, and let prove that $\Pi_S(x+1) > \left[\frac{x+1}{2}\right]$.

Because $\Pi_S(x+1) = \Pi_S(x) + 1$ we have to prove that $\Pi_S(x) + 1 \geq \left[\frac{x+1}{2}\right]$

Therefore $\Pi_S(x) \geq \left[\frac{x+1}{2}\right] - 1$, that is a true relation.

b) x perfect square.

We suppose that $\Pi_S(x) \leq x - [\sqrt{x}]$ is true. Then :

$\Pi_S(x) \le x+1-[\sqrt{x+1}] \Rightarrow \Pi_S(x)+1 \le x+1-[\sqrt{x+1}] \Leftrightarrow \Pi_S(x) \le x-[\sqrt{x+1}]$.
That is a true relation because $[\sqrt{x}] = [\sqrt{x+1}]$. For the left inequality the demonstration is analogous with (a)

2) x prime

a) $x-1$ is not a perfect square.

We suppose that $\Pi_S(x) \le x-[\sqrt{x}]$ is true.
Let prove that $\Pi_S(x+1) \le x+1-[\sqrt{x+1}]$.

In this case we have the following two situations :

(i) If $\Pi_S(x+1) = \Pi_S(x)+1$, then we must prove that :
$\Pi_S(x)+1 \le x+1-[\sqrt{x+1}]$.

Supposing that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true, let show that $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$ or $\Pi_S(x)+1 \ge \left[\frac{x+1}{2}\right]$, therefore $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]-1$ and that results from the hypothesis.

(ii) If $\Pi_S(x+1) = \Pi_S(x)$. We have to prove that $\Pi_S(x) \le x+1-[\sqrt{x+1}]$
Of course this inequality is true. For the left side of the inequality we have to prove that $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$. If we admit $\left[\frac{x}{2}\right] \le \Pi_S(x) < \left[\frac{x+1}{2}\right]$ we obtain that $\Pi_S(x) = \left[\frac{x}{2}\right], x \ne 1$.

According to the *Proposition 2.* this inequality can't be true.

Therefore we have $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$.

Let observe that $x+1$ is not a perfect square, if $x > 3$ is a prime number. For $x = 3$ the inequality is verified by calculus.

3) x is an even composed number. Then :

a) If $x+1$ is a prime.

We know that $\Pi_S(x+1) = \Pi_S(x)+1$. Then supposing $\Pi_S(x) \le x-[\sqrt{x}]$.
We have to prove that $\Pi_S(x+1) \le x+1-[\sqrt{x+1}]$ or $\Pi_S(x) = x-\left[\sqrt{x+1}\right]$.
This is true, because $[\sqrt{x}] = \left[\sqrt{x+1}\right]$.

For the left inequality we have to show $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$,
or $\Pi_S(x)+1 \ge \left[\frac{x+1}{2}\right]$. But $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]-1$, is true.

b) If $x+1$ is an odd composite number, then

(i) If $\Pi_S(x+1) = \Pi_S(x)+1$, the demonstration is the same as at (a).

(ii) If $\Pi_S(x+1) = \Pi_S(x)$, we have to prove that $\Pi_S(x) \le x+1-\left[\sqrt{x+1}\right]$
Obvious.

The left inequality is obvious.

c) $x+1$ perfect square.

Using *Proposition 3* we have only the case $\Pi_S(x+1) = \Pi_S(x)$. Then if we consider to be true the relation $\Pi_S(x) \le x-[\sqrt{x}]$.

Let prove that $\Pi_s(x+1) < x+1-\left[\sqrt{x+1}\right]$.

But $\Pi_s(x) \le x+1-\left[\sqrt{x+1}\right]$ is true.

For the left inequality we suppose that $\Pi_s(x) \ge \left[\frac{x}{2}\right]$ is true. We have to prove that $\Pi_s(x+1) \ge \left[\frac{x+1}{2}\right]$.

Because $\Pi_s(x+1) = \Pi_s(x)$ it results $\Pi_s(x) \ge \left[\frac{x+1}{2}\right]$.

So, we must have $\left[\frac{x}{2}\right] \ge \left[\frac{x+1}{2}\right]$. This is true, because $x+1$ is an odd number.

4) x is an odd composed number.

a) If $x+1$ is even composed number the proof is the same as in (2a).

For the right inequality we have :

(i) If $\Pi_s(x+1) = \Pi_s(x)+1$ and we suppose that $\Pi_s(x) \le x-[\sqrt{x}]$, let to prove that $\Pi_s(x+1) \le x+1-\left[\sqrt{x+1}\right]$.

This relation lead us to $\Pi_s(x) \le x-\left[\sqrt{x+1}\right]$. This is true because $[\sqrt{x}] = \left[\sqrt{x+1}\right]$.

(ii) If $\Pi_s(x+1) = \Pi_s(x)$ the proof is obvious.

b) If $x+1$ is a perfect square.

In this case according to the *Proposition 3* we have only the situation $\Pi_s(x+1) = \Pi_s(x)$. The right sided inequality is obvious and the left side inequality has the same proof as for (2a).

5) If x is a perfect square.

a) If x is a prime and the only situation is that $\Pi_s(x+1) = \Pi_s(x)+1$. The demonstration is obvious.

b) If $x+1$ is a composite number.

For the right inequality we have :

(i) If $\Pi_s(x+1) = \Pi_s(x+1)$, the proof is analogous as in the preceding case.

(ii) If $\Pi_s(x+1) = \Pi_s(x)$ the proof is obvious.

For the left inequality :

If $x+1$ is an odd composite number the relation is obvious.

If $x+1$ is an even composite number then :

if $\Pi_s(x+1) = \Pi_s(x)+1$, the proof is analogous with (a).

if $\Pi_s(x+1) = \Pi_s(x)$ then x can be just an odd perfect square.

We suppose that $\Pi_s(x) \ge \left[\frac{x}{2}\right]$ is true.

To show that $\Pi_s(x) \ge \left[\frac{x+1}{2}\right]$, if we suppose, again, that $\Pi_s(x) < \left[\frac{x+1}{2}\right]$

it results

$$\left[\frac{x}{2}\right] \leq \Pi_S(x) < \left[\frac{x+1}{2}\right], \text{ and we have } \Pi_S = \left[\frac{x}{2}\right].$$

Proposition 5.

$$\lim_{n\to\infty} [\ \Pi_S(2n) - \Pi_S(n)\ ] = \infty.$$

Proof.

According to the *Proposition 4* we have :

$$\Pi_S(n) \leq n - \left[\sqrt{n+1}\right] < n < \Pi_S(2n) \Rightarrow$$

$$\Pi_S(2n) - \Pi_S(n) > \left[\sqrt{n+1}\right] \text{ and } \lim_{n\to\infty} \left[\sqrt{n+1}\right] = \infty.$$

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ON THREE NUMERICAL FUNCTIONS

by

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In this paper we define the numerical functions φ_S , φ_S^*, ω_S and we prove some properties of these functions.

1. Definition. If S(n) is the Smarandache function, and (m, n) is the greatest common divisor of m and n, then the functions φ_S , φ_S^* and ω_S are defined on the set **N*** of the positive integers, with values in the set **N** of all the non negative integers, such that:

$\varphi_S(x) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \leq x, (S(m), x) = 1\}$

$\varphi_S^*(x) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \leq x, (S(m), x) \neq 1\}$

$\omega_S(x) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \leq x, \text{and } S(m) \text{ divides } x\}$.

From this definition it results that:

$$\varphi_S(x) + \varphi_S^*(x) = x \text{ and } \omega_S(x) \leq \varphi_S^*(x) \qquad (1)$$

for all $x \in \mathbf{N}^*$.

2. Proposition. For every prime number $p \in \mathbf{N}^*$ we have

$\varphi_S(p) = p - 1 = \varphi(p)$, $\varphi_S(p^2) = p^2 - p = \varphi(p^2)$

where φ is Euler's totient function.

Proof. Of course, if p is a prime then for all integer a satisfying $0 < a \leq p - 1$ we have $(S(a), p) = 1$, because $S(a) \leq a$. So, if we note $M_1(x) = \{m \in \mathbf{N}^* / 0 < m \leq p, (S(m), p) = 1\}$ then $a \in M_1(p)$.

At the same time, because $S(p) = p$, it results that $(S(p), p) = p \neq 1$ and so $p \notin M_1(p)$.

Then we have $\varphi_S(p) = p - 1 = \varphi(p)$.

The positive integers a, not greater than p^2 and not belonging to the set $M_1(p^2)$ are: $p, 2p, \ldots, (p-1)p, p^2$.

For $p = 2$ this assertion is evidently true, and if p is an odd prime number then for all $h < p$ it results $S(h \cdot p) = p$.

Now, if $m < p^2$ and $m \neq hp$ then $(S(m), p^2) = 1$. Indeed, if for $m = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, $q_i \neq p$ we have $(S(m), p^2) \neq 1$, then it exists a divisor q^α of m such that $S(m) = S(q^\alpha) = q(\alpha - i_\alpha)$, with $i_\alpha \in \left[0; \left[\frac{\alpha - 1}{q}\right]\right]$.

From $(q(\alpha - i_\alpha), p^2) \neq 1$ it results $(q(\alpha - i_\alpha), p) \neq 1$ and because $q \neq p$ it results $(\alpha - i_\alpha, p) \neq 1$, so $(\alpha - i_\alpha, p) = p$. But p does not divide $\alpha - i_\alpha$ because $\alpha < p$.

Indeed, we have:

$$q^\alpha < p^2 \Leftrightarrow \alpha < 2\log_q p \leq 2 \cdot \frac{p}{2} = p$$

because we have:

$$\log_q p \leq \frac{p}{2} \text{ for } q \geq 2 \text{ and } p \geq 3.$$

So,

$$\varphi_S(p^2) = p^2 - \text{Card}\{1 \cdot p, 2 \cdot p, \ldots, (p-1)p, p^2\} = p^2 - p = \varphi(p^2).$$

3. Proposition. For every $x \in \mathbf{N}^*$ we have:

$$\varphi_S(x) \leq x - \tau(x) + 1$$

where $\tau(x)$ is the number of the divisors of x.

Proof. From (1) it results that $\varphi_S(x) = x - \varphi_S^*(x)$, and of course, from the definition of φ_S^* and τ it results $\varphi_S^*(x) \geq \tau(x) - 1$. Then $\varphi_S(x) \leq x - \tau(x) + 1$. Particularly, if x is a prime then $\varphi_S(x) \leq x - 1$, because in this case $\tau(x) = 2$.

If x is a composite number, it results that $\varphi_S(x) \leq x - 2$.

4. Proposition. If $p < q$ are two consecutive primes then :

$$\varphi_S(pq) = \varphi(pq).$$

Proof. Evidently, $\varphi(pq) = (p - 1)(q - 1)$ and

$$\varphi_S(pq) = \text{Card}\{m \in \mathbf{N}^* / 0 < m \leq pq, (S(m), pq) = 1\}.$$

Because p and q are consecutive primes and $p < q$ it results that the multiples of p and q which are not greater than pq are exactly given by the set:

$$M = \{p, 2p, \ldots, p^2, (p+1)p, \ldots, (q-1)p, qp, q, 2q, \ldots, (p-1)q\}.$$

These are in number of $p + q - 1$.

Evidently, $(S(m), pq) \neq 1$ for $m \in \{p, 2p, \ldots, (p-1)p, p^2, q, 2q, \ldots, (p-1)q\}$.

Let us calculate $S(m)$ for $m \in \{(p+1)p, (p+2)p, \ldots, (q-1)p\}$.

Evidently, $(p+i, p) = 1$ for $1 \leq i \leq q-p-1$, and so $[p+i, p] = p(p+i)$.

It results that $S(p(p+i)) = S([p, p+i]) = \max\{S(p), S(p+i)\} = S(p)$.

Indeed, to estimate $S(p+i)$ let $p+i = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_h^{\alpha_h} < q < 2p$.

Then $p_1^{\alpha_1} < p, p_2^{\alpha_2} < p, \ldots, p_h^{\alpha_h} < p$.

It results that:

$S(p+i) = S(p_j^{\alpha_j}) < S(p)$, for some $j = \overline{1,h}$.

It results that:

$(S(p(p+i)), pq) = (p, pq) = p \neq 1$.

In the following we shall prove that if $0 < m \leq pq$ and m is not a multiple of p or q then $(S(m), pq) = 1$.

It is said that if $m < p^2$ is not a multiple of p then $(S(m), p) = 1$.

If $m \leq q^2$ is not a multiple of q then it results also $(S(m), q) = 1$.

Now, if $m < p^2$ (and of course $m < q^2$) is not a multiple either of p and q then from $(S(m), p) = 1$ and $(S(m), q) = 1$ it results $(S(m), pq) = 1$.

Finally, for $p^2 < m < pq < q^2$, with m not a multiple either of p and q, if the decomposition of m into primes is $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_S^{\alpha_S}$ then $S(m) = S(p_k^{\alpha_k}) < S(p) = p$ so $(S(m), p) = 1$.

Analogously, $(S(m), q) = 1$, and so $(S(m), pq) = 1$.

Consequently,

$$\varphi_S(pq) = pq - p - q + 1 = \varphi_S(pq).$$

5. Proposition

(i) If $p > 2$ is a prime number then $\omega_S(p) = 2$, $\omega_S(p^2) = p$.

(ii) If x is a composite number then $\omega_S(x) \geq 3$.

Proof. From the definition of the function ω_S it results that $\omega_S(p) = 2$.

If $1 \le m \le p^2$, from the condition that S(m) divides p^2 it results m = 1 or m = kp, with

$k \le p - 1$, so :

$m \in \{1, p, 2p, \ldots, (p-1)p\}$ and $\omega_S(p^2) = p$.

If x is a composite number, let p be one of its prime divisors.

Then, of course, $1, p, 2p \in \{m / 0 < m \le x\}$.

If $p > 3$ then :

S(1) = 1 divides x, S(p) = p divides x and S(2p) = S(p) = p divides x.

It rezults $\omega_S(x) \ge 3$.

If $x = 2^\alpha$, with $\alpha \ge 2$ then :

S(1) = 1 divides x, S(2) = 2 divides x and S(4) = 4 divides x ,

so we have also $\omega_S(x) \ge 3$.

6. Proposition. For every positive integer x we have :

$$\omega_S(x) \le x - \varphi(x) + 1. \qquad (2)$$

Proof. We have $\varphi(x) = x$ - Card A, when

$A = \{m / 0 < m \le x, (m, x) \ne 1\}$.

Evidently, the inequality (2) is valid for all the prime numbers.

If x is a composite number it results that at least a proper divisor of m is also a divisor of S(m) and of x. So $(m, x) \ne 1$ and consequently $m \in A$.

So, $\{m / 0 < m \le x, S(m) \text{ divides } x\} \subset A \cup \{1\}$ and it results that :

Card $\{m / 0 < m \le x, S(m) \text{ divides } x\} \le$ Card A - 1, or

$\omega_S(x) \le 1 +$ Card A,

and from this it results (2).

7. Proposition. The equation $\omega_S(x) = \omega_S(x + 1)$ has not a solution between the prime numbers.

Proof. Indeed, if x is a prime then $\omega_S(x) = 2$ and because x + 1 is a composite number it results $\omega_S(x + 1) \ge 3$.

Let us observe that the above equation has solutions between the primes. For instance, $\omega_s(35) = \omega_s(36) = 11$.

8. Proposition. The function $\varphi_s(x)$ has all the primes as local maximal points.

Proof. We have $\varphi_s(p) = p - 1$, $\varphi_s(p - 1) \leq p - 3 < \varphi_s(p)$ and $\varphi_s(p + 1) \leq \varphi_s(p)$, because $p + 1$ being a composite number has at least two divisors.

Let us mention now the following unsolved problems:

(UP_1) There exists $x \in \mathbf{N}^*$ such that $\varphi_s(x) < \varphi(x)$.

(UP_2) For all $x \in \mathbf{N}^*$ is valid the inequality

$$\omega_s(x) \geq \tau(x)$$

where $\tau(x)$ is the number of the divisors of x.

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THE MONOTONY OF SMARANDACHE FUNCTIONS OF FIRST KIND

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Smarandache functions of first kind are defined in [1] thus:

$$S_n: N^* \to N^*, \quad S_1(k) = 1 \text{ and } S_n(k) = \max_{1 \le j \le r} \{S_{p_j}(i_j k)\},$$

where $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ and S_{p_j} are functions defined in [4].

They $\sum_1$ - standardise $(N^*, +)$ in $(N^*, \le, +)$ in the sense that

$$\sum_1: \quad \max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

for every $a, b \in N^*$ and $\sum_2$ - standardise $(N^*, +)$ in $(N^*, \le, \cdot)$ by

$$\sum_2: \quad \max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) \cdot S_n(b), \text{ for every } a, b \in N^*$$

In [2] it is prooved that the functions S_n are increasing and the sequence $\{S_{p^i}\}_{i \in N^*}$ is also increasing. It is also proved that if p, q are prime numbers, then

$$p \cdot i < q \Rightarrow S_{p^i} < S_q \text{ and } i < q \Rightarrow S_i < S_q,$$

where $i \in N^*$.

It would be used in this paper the formula

$$S_p(k) = p(k - i_k), \text{ for same } i_k \text{ satisfying } 0 \le i_k \le \left[\frac{k-1}{p}\right], \text{ (see [3])} \qquad (1)$$

1. Proposition. *Let p be a prime number and $k_1, k_2 \in N^*$. If $k_1 < k_2$ then $i_{k_1} \le i_{k_2}$, where i_{k_1}, i_{k_2} are defined by* (1).

Proof. It is known that $S_p: \mathbf{N}^* \to \mathbf{N}^*$ and $S_p(k) = pk$ for $k \le p$. If $S_p(k) = mp^\alpha$ with $m, \alpha \in \mathbf{N}^*, (m, p) = 1$, there exist α consecutive numbers:

$$n, n+1, \ldots, n+\alpha-1 \quad \text{so that}$$
$$k \in \{n, n+1, \ldots, n+\alpha-1\} \quad \text{and}$$
$$S_p(n) = S_p(n+1) = \cdots = S(n+\alpha-1),$$

this means that S_p is stationed the $\alpha-1$ steps $(k \to k+1)$.

If $k_1 < k_2$ and $S_p(k_1) = S_p(k_2)$, because $S_p(k_1) = p(k_1 - ik_1)$, $S_p(k_2) = p(k_2 - ik_2)$ it results $i_{k_1} < i_{k_2}$.

If $k_1 < k_2$ and $S_p(k_1) < S_p(k_2)$, it is easy to see that we can write:

$$i_{k_1} = \beta_1 + \sum_{\substack{\alpha \\ mp^\alpha < S_p(k_1)}} (\alpha - 1), \quad \text{where} \quad \beta_1 = 0 \text{ for } S_p(k_1) \neq mp^\alpha, \quad \text{if } S_p(k_1) = mp^\alpha$$

then $\beta_1 \in \{0,1,2,\ldots,\alpha-1\}$
and

$$i_{k_2} = \beta_2 + \sum_{\substack{\alpha \\ mp^\alpha < S_p(k_2)}} (\alpha - 1), \quad \text{where} \quad \beta_2 = 0 \text{ for } S_p(k_2) \neq mp^\alpha, \text{ if } S_p(k_2) = mp^\alpha \text{ then}$$

$\beta_2 \in \{0,1,2,\ldots,\alpha-1\}$.

Now is obviously that $k_1 < k_2$ and $S_p(k_1) < S_p(k_2) \Rightarrow i_{k_1} \leq i_{k_2}$. We note that, for $k_1 < k_2$, $i_{k_1} = i_{k_2}$ iff $S_p(k_1) < S_p(k_2)$ and $\{mp^\alpha \mid \alpha > 1 \text{ and } mp^\alpha \leq S_p(k_1)\} = \{mp^\alpha \mid \alpha > 1 \text{ and } mp^\alpha < S_p(k_2)\}$

2. Proposition. *If p is a prime number and $p \geq 5$, then $S_p > S_{p-1}$ and $S_p > S_{p+1}$.*

Proof. Because $p-1 < p$ it results that $S_{p-1} < S_p$. Of course $p+1$ is even and so:

(i) if $p+1 = 2^i$, then $i > 2$ and because $2i < 2^i - 1 = p$ we have $S_{p+1} < S_p$.

(ii) if $p+1 \neq 2^i$, let $p+1 = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$, then $S_{p+1}(k) = \max_{1 \leq j \leq r} \{S_{p_j^{i_j}}(k)\} = S_{p_m^{i_m}}(k) =$

$= S_{p_m}(i_m \cdot k)$.

Because $p_m \cdot i_m \leq p_m^{i_m} \leq \frac{p+1}{2} < p$ it results that $S_{p_m^{i_m}}(k) < S_p(k)$ for $k \in N^*$, so that $S_{p+1} < S_p$.

3. Proposition. *Let p,q be prime numbers and the sequences of functions*

$$\{S_{p^i}\}_{i \in N^*}, \quad \{S_{q^j}\}_{j \in N^*}$$

If $p < q$ and $i \leq j$, then $S_{p^i} < S_{q^j}$.

Proof. Evidently, if $p < q$ and $i \leq j$, then for every $k \in N^*$

$$S_{p^i}(k) \leq S_{p^j}(k) < S_{q^j}(k)$$

so, $$S_{p^i} < S_{q^j}$$

4. Definition. *Let p,q be prime numbers. We consider a function S_{q^j}, a sequence of functions $\{S_{p^i}\}_{i \in N^*}$, and we note:*

$$i_{(j)} = \max_i \left\{ i \mid S_{p^i} < S_{q^j} \right\}$$

$$i^{(j)} = \min_{i}\left\{i \middle| S_{q^j} < S_{p^i}\right\},$$

then $\{k \in N \middle| i_{(j)} < k < i^{(j)}\} = \Delta_{p^j(q^j)} = \Delta_{i(j)}$ *defines the interference zone of the function* S_{q^j} *with the sequence* $\left\{S_{p^i}\right\}_{i \in N^*}$.

5. ***Remarque.***

a) If $S_{q^j} < S_{p^i}$ for $i \in \mathbf{N}^*$, then now exists $i_{(j)}$ and $i^{(j)} = 1$, and we say that S_{q^j} is separately of the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$.

b) If there exist $k \in \mathbf{N}^*$ so that $S_{p^k} < S_{q^j} < S_{p^{k+1}}$, then $\Delta_{p^j(q^j)} = \varnothing$ and say that the function S_{q^j} does not interfere with the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$.

6. **Definition**. *The sequence* $\{x_n\}_{n \in N^*}$ *is generaly increasing if*

$$\forall n \in N^* \ \exists m_0 \in N^* \text{ so that } x_m \geq x_n \text{ for } m \geq m_0.$$

7. ***Remarque***. If the sequence $\{x_n\}_{n \in N^*}$ with $x_n \geq 0$ is generaly increasing and boundled, then every subsequence is generaly increasing and boundled.

8. **Proposition**. *The sequence* $\{S_n(k)\}_{n \in N^*}$, *where* $k \in N^*$, *is in generaly increasing and boundled.*

Proof. Because $S_n(k) = S_{n^k}(1)$, it results that $\{S_n(k)\}_{n \in N^*}$ is a subsequence of $\{S_m(1)\}_{m \in N^*}$.

The sequence $\{S_m(1)\}_{m \in N^*}$ is generaly increasing and boundled because:

$$\forall m \in N^* \ \exists t_0 = m! \text{ so that } \forall t \geq t_0 \ S_t(1) \geq S_{t_0}(1) = m \geq S_m(1).$$

From the remarque 7 it results that the sequence $\{S_n(k)\}_{n \in N^*}$ is generaly increasing boundled.

9. **Proposition**. *The sequence of functions* $\{S_n\}_{n \in N^*}$ *is generaly increasing boundled.*

Proof. Obviously, the zone of interference of the function S_m with $\{S_n\}_{n \in N^*}$ is the set

$$\Delta_{n(m)} = \{k \in N^* \middle| n_{(m)} < k < n^{(m)}\} \text{ where}$$

$$n_{(m)} = \max\{n \in N^* \middle| S_n < S_m\}$$

$$n^{(m)} = \min\{n \in N^* \middle| S_m < S_n\}.$$

The interference zone $\Delta_{n(m)}$ is nonemty because $S_m \in \Delta_{n(m)}$ and finite for $S_1 \le S_m \le S_p$, where p is one prime number greater than m.

Because $\{S_n(1)\}$ is generaly increasing it results:

$$\forall m \in N^* \ \exists t_0 \in N^* \text{ so that } S_t(1) \ge S_m(1) \text{ for } \forall t \ge t_0.$$

For $r_0 = t_0 + n^{(m)}$ we have

$$S_r \ge S_m \ge S_m(1) \text{ for } \forall r \ge r_0,$$

so that $\{S_n\}_{n \in N^*}$ is generaly increasing boundled.

10. *Remarque.*

a) For $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ are posible the following cases:

1) $\exists\, k \in \{1,2,...,r\}$ so that

$$S_{p_j^{i_j}} \le S_{p_k^{i_k}} \text{ for } j \in \{1,2,...,r\},$$

then $S_n = S_{p_k^{i_k}}$ and $p_k^{i_k}$ is named the dominant factor for n.

2) $\exists\, k_1, k_2, ..., k_m \in \{1,2,...,r\}$ so that :

$\forall t \in \overline{1,m} \quad \exists q_t \in N^*$ so that $S_n(q_t) = S_{p_{k_t}^{i_{k_t}}}(q_t)$ and

$$\forall\, l \in N^* \quad S_n(l) = \max_{1 \le t \le m} \left\{ S_{p_{k_t}^{i_{k_t}}}(l) \right\}.$$

We shall name $\{p_{k_t}^{i_{k_t}} \mid t \in \overline{1,m}\}$ the active factors, the others wold be name passive factors for n.

b) We consider

$$N_{p_1 p_2} = \{n = p_1^{i_1} \cdot p_2^{i_2} \mid i_1, i_2 \in N^*\}, \text{ where } p_1 < p_2 \text{ are prime numbers.}$$

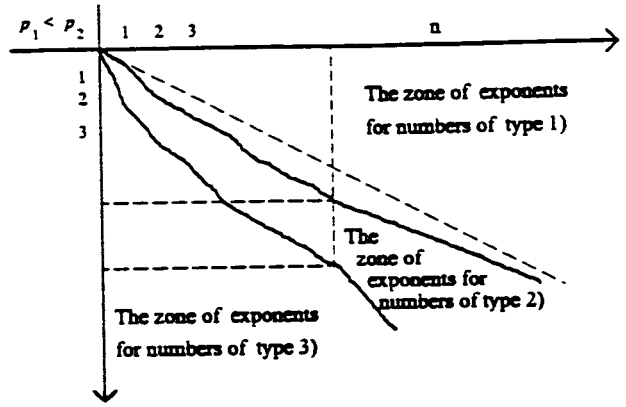
For $n \in N_{p_1 p_2}$ appear the following situations:

1) $i_1 \in (0, i_1^{(i_2)}]$, this means that $p_1^{i_1}$ is a pasive factor and $p_2^{i_2}$ is an active factor.

2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ this means that $p_1^{i_1}$ and $p_2^{i_2}$ are active factors.

3) $i_1 \in [i_1^{(i_2)}, \infty)$ this means that $p_1^{i_1}$ is a active factor and $p_2^{i_2}$ is a pasive factor.

For $p_1 < p_2$ the repartion of exponents is represently in following scheme:


$p_1 < p_2$
1 2 3
n
1
2
3
The zone of exponents for numbers of type 1)
The zone of exponents for numbers of type 2)
The zone of exponents for numbers of type 3)

For numbers of type 2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ and $i_2 \in (i_{2(i_1)}, i_2^{(i_1)})$

c) I consider that

$$N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \,|\, i_1, i_2, i_3 \in N^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1,2,3$ this means that $p_j^{i_j}$ is active factor.

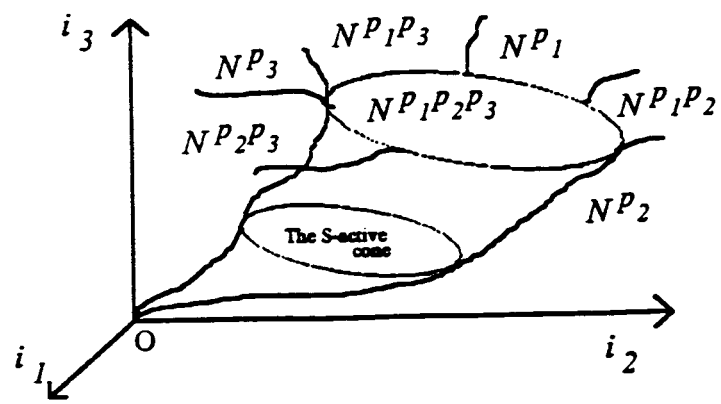
2) $n \in N^{p_j p_k}, j \neq k;\ j,k \in \{1,2,3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1 p_2 p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1 p_2 p_3}$ is named the S-active cone for $N_{p_1 p_2 p_3}$.

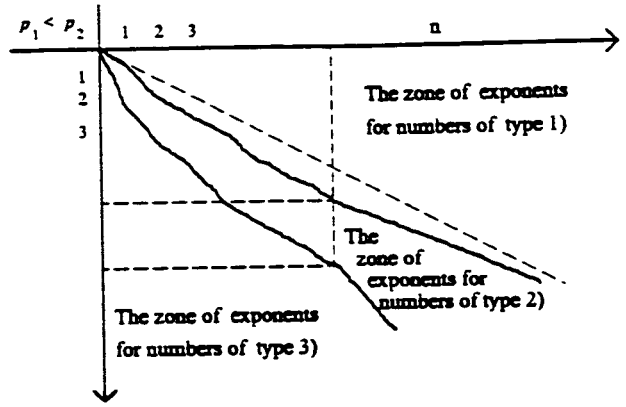
Obviously

$$N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} \,|\, i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i_j)}, i_k^{(ij)}) \text{ where } j \neq k; j,k \in \{1,2,3\}\}.$$

The repartision of exponents is represented in the following scheme:



For $p_1 < p_2$ the repartion of exponents is represently in following scheme:


$p_1 < p_2$
1 2 3
n
1
2
3
The zone of exponents for numbers of type 1)
The zone of exponents for numbers of type 2)
The zone of exponents for numbers of type 3)

For numbers of type 2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ and $i_2 \in (i_{2(i_1)}, i_2^{(i_1)})$

c) I consider that

$$N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \,|\, i_1, i_2, i_3 \in N^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1,2,3$ this means that $p_j^{i_j}$ is active factor.

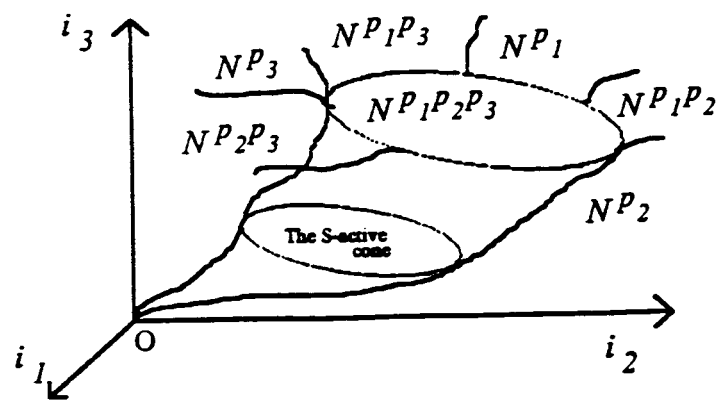
2) $n \in N^{p_j p_k}, j \neq k;\ j,k \in \{1,2,3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1 p_2 p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1 p_2 p_3}$ is named the S-active cone for $N_{p_1 p_2 p_3}$.

Obviously

$$N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} \,|\, i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i_j)}, i_k^{(ij)}) \text{ where } j \neq k; j,k \in \{1,2,3\}\}.$$

The repartision of exponents is represented in the following scheme:



d) Generaly, I consider $N_{p_1 p_2 \ldots p_r} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot \cdots \cdot p_r^{i_r} \mid i_1, i_2, \ldots, i_r \in N^*\}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers.

On $N_{p_1 p_2 \ldots p_r}$ exist the following relation of equivalence:

$$n \rho m \Leftrightarrow n \text{ and } m \text{ have the same active factors.}$$

This have the following clases:

- $N^{p_{j_1}}$, where $j_1 \in \{1,2,\ldots,r\}$.

$n \in N^{p_{j_1}} \Leftrightarrow n$ hase only $p_{j_1}^{i_{j_1}}$ active factor

- $N^{p_{j_1} p_{j_2}}$, where $j_1 \neq j_2$ and $j_1, j_2 \in \{1,2,\ldots,r\}$.

$n \in N^{p_{j_1} p_{j_2}} \Leftrightarrow n$ has only $p_{j_1}^{i_{j_1}}$, $p_{j_2}^{i_{j_2}}$ active factors.

. . . . . .

$N^{p_1 p_2 \cdots p_r}$ wich is named S-active cone.

$N^{p_1 p_2 \cdots p_r} = \{n \in N_{p_1 p_2 \cdots p_r} \mid n \text{ has } p_1^{i_1}, p_2^{i_2}, \ldots, p_r^{i_r} \text{ active factors}\}$.

Obviously, if $n \in N^{p_1 p_2 - p_r}$, then $i_k \in (i_{k(i_j)}, i_k^{(i_j)})$ with $k \neq j$ and $k, j \in \{1,2,\ldots,r\}$.

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THE SMARANDACHE NEAR-TO-PRIMORIAL (S.N.T.P.) FUNCTION

by

M. R. Mudge

Definition A.

The PRIMORIAL Function, p*, of a prime number, p, is defined be the product of the prime numbers less than or equal to p. e.g. 7* = 2·3·5·7 = 210 similarly 11* = 2310. A number, q, is said to be near to prime if and only if *either* q+1 *or* q-1 are primes it is said to be the mean-of-a-prime-pair if and only if *both* q+1 *and* q-1 are prime.

p such that p* is near to prime: 2, 7, 13, 37, 41, 53, 59, 67, 71, 79, 83, 89, ...

p such that p* is mean-of-a-prime-pair: 3, 5, 11, 31, ...

TABLE I

p	2	3	5	7	11	13
p*-1	1	5	29p	209=11·19	2309p	30029p
p*	2	6	30	210	2310	30030
p*+1	3	7	31p	211p	2311p	30031=59·509

Definition B.

The SMARANDACHE Near-To-Primorial Function, SPr(n), is defined as the smallest prime p such that either p* or p* ± 1 is divisible by n.

n	1	2	3	4	5	6	7	8	9	10	11	...59...
SPr(n)	2	2	2	5	3	3	3	5	?	5	11	13

Questions relating to this function include:

1. Is SPr(n) defined for all positive integers n ?
2. What is the distribution of values of SPr(n) ?
3. Is this problem fundamentally altereted by replacing p* ± 1 by p* ± 3, 5, ...

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A Note on the Smarandache Near-To-Primorial Function

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In a brief paper passed on to the author[1], Michael R. Mudge used the definition of the Primorial function:

Definition: For p any prime, the Primorial function of p, p* is the product of all prime numbers less than or equal to p.

Examples:

3* = 2 * 3 = 6
11* = 2 * 3 * 5 * 7 * 11 = 2310

To define the Smarandache Near-To-Primorial Function SPr(n)

Definition: For n a positive integer, the Smarandache Near-To-Primorial Function SPr(n) is the smallest prime p such that either p* or p* + 1 or p* - 1 is divisible by n.

A table of initial values is also given

n	1	2	3	4	5	6	7	8	9	10	11 . . . 59
SPr(n)	2	2	2	5	3	3	3	5	?	5	11 . . . 13

and the following questions posed:

1) Is SPr(n) defined for all positive integers n?
2) What is the distribution of values of SPr(n)?
3) Is this problem fundamentally altered by replacing p* $\pm$ 1 by p* $\pm$ k for k = 3,5,...?

The purpose of this paper is to address these questions.

We start with a simple but important result that is presented in the form of a lemma.

Lemma 1: If the prime factorization of n contains more than one instance of a prime as a factor, then n cannot divide q* for q any prime.

Proof: Suppose that n contains at least one prime factor to a power greater than one, for reference purposes, call that prime p1. The list of prime factors of n contains a largest

prime and we can call that prime p2. If we choose another arbitrary prime q, there are two cases to consider.

Case 1: $q < p2$. Then p2 cannot divide q*, as q* contains no instances of p2 by definition.

Case 2: $q \geq p2$. In this case, each prime factor of n will divide q*, but since p1 appears only once in q*, $p1^2$ cannot divide q*. Therefore, n cannot divide q* as well. □

We are now in a position to answer the first question.

Theorem 1: If n contains more than one instance of 2 as a factor, then SPr(n) does not exist.

Proof: Choose n to be a number having more than one instance of 2 as a factor. By lemma 1, there is no prime q such that n divides q*. Furthermore, since 2 is a prime, q* is always even. Therefore, $q^* \pm 1$ is always odd and n cannot evenly divide it. □

The negative answer to the first question also points out two errors in the Mudge table. SPr(4) and SPr(8) do not exist, and an inspection of the given values verifies this. The Primorial of 5 is 2*3*5 = 30 and no element in the set { 29,30,31 } is evenly divisible by 4.

By definition, the range of SPr(n) is a set of prime numbers. The obvious question is then whether the range of SPr(n) is in fact the set of all prime numbers, and we state the answer as a theorem.

Theorem 2: The range of SPr(n) is the set of all prime numbers.

Proof: The first few values are by inspection.

SPr(1) = 2, SPr(5) = 3, SPr(10) = 5

Choose an arbitrary prime $p > 5$ and construct the number p* - 1. Obviously, p* - 1 divides p* -1. It is also clear that there is no prime $q < p$ such that q*, $q^* + 1$ or $q^* - 1$ is divisible by p* -1. Therefore, SPr(p* - 1) = p and p is in the range of SPr(n). □

Which answers the second question posed by M. Mudge.

It is easy to establish an algorithmic process to determine if SPr(n) is defined for values of n containing more than one instance of a prime greater than 2.

The first step is to prove another lemma.

Lemma 2: If n contains a prime p that appears more than once as a factor of n, and q is any prime $q \geq p$, then n does not divide $q^* \pm 1$.

Proof: Let n, p and q have the stated properties. Clearly, p divides q^* and since q is greater than 1, p cannot divide $q^* \pm 1$, forcing the conclusion that n cannot divide $q^* \pm 1$ as well. Combining this with lemma 1 gives the desired result. □

Corollary: If n contains some prime p more than once as a factor and SPr(n) exists, then the prime q such that n divides $q^* \pm 1$ must be less than p.

Proof: Clear. □

The next lemma deals with some of the instances where SPr(n) is defined.

Lemma 3: If $n = p_1 p_2 \ldots p_k$, where $k \geq 1$ and all p_i are primes, then SPr(n) is defined.

Proof: Let q denote the largest prime factor of n. By definition, q^* contains one instance of all primes less than or equal to q, so n must divide q^*. Given the existence of one such number, there must also be a minimal one. □

Combining all previous results, we can create a simple algorithm that can be used to determine if SPr(n) exists for any positive integer n.

Input: A positive integer n.
Output: Yes, if SPr(n) exists, No otherwise.

Step 1: Factor n into prime factors, $p_1 p_2 \ldots p_k$.
Step 2: If all primes appear to the first power, terminate with the message "Yes".
Step 3: If 2 appears to a power greater than 1, terminate with the message "No".
Step4: Set q = 2, the smallest prime.
Step 5: Compute $q^* + 1$ and $q^* - 1$.
Step 6: If n divides $q^* + 1$ or $q^* - 1$, terminate with the message "Yes".
Step 7: Increment q to the next largest prime.
Step 8: If $q \geq p$, terminate with the message "No".
Step 9: Goto step 5.

And this algorithm can be used to resolve the question mark in the Mudge table. Since 9 does not divide $2^* \pm 1$, SPr(9) is not defined. Furthermore, 3 to any power greater than 2 also cannot divide $2^* \pm 1$, so the conclusion is stronger in that SPr(n) is not defined for n any power of 3 greater than 3.

Note that modifications of this algorithm could be made so that it also returns the value of SPr(n) when defined.

These conclusions can be used to partially answer the third question. The conclusion of lemma 3 concerning all prime factors to the first power is unaffected. However, if $q \geq 3$ and q prime, then $q^* \pm 3$ is also divisible by 3, making solutions possible for higher powers of 3. Such results do indeed occur, as

$$3^* + 3 = 9$$

so that the modified SPr(9) = 9.

Reference

1. **The Smarandache Near-To-Primorial Function**, personal correspondence by Michael R. Mudge.

PRIMES BETWEEN CONSECUTIVE SMARANDACHE NUMBERS

by

G. Suggett

I assume that the range between S(n) and S(n+1) should be interpreted as including the endpoints ? If one is looking for cases in which there are no primes in the open interval between the two consecutive values, then the range of exceptions is much larger, including n = 1, 2, 3, 4, 5, 9, 14, 15, ... Using the closed interval gives a much smaller list of exceptions, starting, as you state, with n = 224. I have confirmed that the next value is n = 2057, but to go further on a systematic basis would be far too time-consuming. However, taking the hint about prime pairs, I have found the following:

Associated with the prime pair (101, 103): 265225, 265226
Associated with the prime pair (107, 109): 67697937, 67697938
Associated with the prime pair (149, 151): 843637, 843638
Associated with the prime pair (461, 463): 24652435, 24652436
Associated with the prime pair (521, 523): 35558770, 35558771
Associated with the prime pair (569, 571): 46297822, 46297823
Associated with the prime pair (821, 823): 138852445, 138852446
Associated with the prime pair (857, 859): 157906534, 157906535
Associated with the prime pair (881, 883): 171531580, 171531581
Associated with the prime pair (1061, 1063): 299441785, 299441786
Associated with the prime pair (1301, 1303): 551787925, 551787926
Associated with the prime pair (1697, 1699): 1223918824, 1223918825
Associated with the prime pair (1721, 1723): 1276553470, 1276553471
Associated with the prime pair (1787, 1789): 5108793239997, 5108793239998
Associated with the prime pair (1871, 1873): 6138710055036, 6138710055037
Associated with the prime pair (1877, 1879): 1655870629, 1655870630
Associated with the prime pair (1949, 1951): 1853717287, 1853717288
Associated with the prime pair (1997, 1999): 1994004499, 1994004500
Associated with the prime pair (2081, 2083): 2256222280, 2256222281
Associated with the prime pair (2111, 2113): 9945866761776, 9945866761777
Associated with the prime pair (2237, 2239): 2802334639, 2802334640
Associated with the prime pair (2381, 2383): 3378819955, 3378819956

Associated with the prime pair (2657, 2659): 4694666584, 4694666585
Associated with the prime pair (2729, 2731): 5086602202, 5086602203
Associated with the prime pair (2801, 2803): 5499766300, 5499766301
Associated with the prime pair (3251, 3253): 55912033969191, 55912033969192
Associated with the prime pair (3257, 3259): 8645559934, 8645559935
Associated with the prime pair (3461, 3463): 10373399185, 10373399186
Associated with the prime pair (3557, 3559): 11260501609, 11260501610
Associated with the prime pair (3581, 3583): 11489910655, 11489910656
Associated with the prime pair (3671, 3673): 90891127331586, 90891127331587
Associated with the prime pair (3917, 3919): 15036031219, 15036031220
Associated with the prime pair (3929, 3931): 15174611302, 15174611303
Associated with the prime pair (4001, 4003): 16024009000, 16024009001
Associated with the prime pair (4127, 4129): 145169740720152, 145169740720153
Associated with the prime pair (4217, 4219): 18761158894, 18761158895
Associated with the prime pair (4241, 4243): 19083231940, 19083231941
Associated with the prime pair (4421, 4423): 21617036545, 21617036546
Associated with the prime pair (4517, 4519): 23055716569, 23055716570
Associated with the prime pair (4547, 4549): 213896677247667, 213896677247668
Associated with the prime pair (4649, 4651): 25136152762, 25136152763
Associated with the prime pair (4721, 4723): 26321940220, 26321940221
Associated with the prime pair (5009, 5011): 31437871492, 31437871493
Associated with the prime pair (5021, 5023): 31664313895, 31664313896
Associated with the prime pair (5099, 5101): 338226861243825, 338226861243826
Associated with the prime pair (6089, 6091): 56466627682, 56466627683
Associated with the prime pair (6197, 6199): 59524353949, 59524353950
Associated with the prime pair (6569, 6571): 70898343322, 70898343323
Associated with the prime pair (6701, 6703): 75258100075, 75258100076
Associated with the prime pair (6869, 6871): 81060670597, 81060670598
Associated with the prime pair (7457, 7459): 103706773384, 103706773385
Associated with the prime pair (7589, 7591): 109311364057, 109311364058
Associated with the prime pair (7757, 7759): 116731835059, 116731835060

and so on. I am reaching the limits of my computational power, but with no obvious end in sight to the list. Do you have a copy of Radu's proof that the set is finite ? Does it give an upper bound on the values in the set ? I am intrigued.

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Introducing the SMARANDACHE-KUREPA and SMARANDACHE-WAGSTAFF Functions

by

M. R. Mudge

Definition A.

The left-factorial function is defined by D.Kurepa thus:

$$!n = 0! + 1! + 2! + 3! + \ldots + (n-1)!$$

whilst S.S.Wagstaff prefers:

$$B_n = !(n+1) - 1 = 1! + 2! + 3! + \ldots + n!$$

The following properties should be observed:

(i) $!n$ is only divisible by n when $n = 2$.
(ii) 3 is a factor of B_n if n is greater than 1.
(iii) 9 is a factor of B_n if n is greater than 4.
(iv) 99 is a factor of B_n if n is greater than 9.

There are no other such cases of divisibility ob B_n for n less than a thousand.

The tabulated values of these two functions together with their prime factors begin:

TABLE I.

n	!n	B_n
1	1	1
2	2	3
3	4=2·2	9=3·3
4	10=2·5	33=3·11
5	34=2·17	153=3·3·17
6	154=2·7·11	873=3·3·97
7	8742·19·23	5913=3·3·3·3·73
8	5914=2·2957	46233=3·3·11·467
9	46234=2·23117	409113=3·3·131·347
10	409114=2·204557	

"Intuitive Thought": There appear to be a disproportionate (unexpectedly high) number of large primes in this table?

Definition B.

For prime p not equal to 3 define the SMARANDACHE-KUREPA Function, SK(p), as the smallest integer such that !SK(p) is divisible by p.
For prime p not equal to 2 or 5 define the SMARANDACHE-WAGSTAFF Function, SW(p), as the smallest integer such that $B_{SW(p)}$ is divisible by p.

The tabulation of these two functions begins:

TABLE II.

p	2	3	5	7	11	13	17	19	23	131
SK(p)	2	*	4	6	6	?	5	7	7	?
SW(p)	*	2	*	?	4	?	5	?	?	9

Where the entry * denotes that the value is not defined and the entry ? denotes not avaible from TABLE I above.

Some unanswered questions:

1. Are there other (*) - entries i.e. undefined values in the above table.
2. What is the distribution function of integers in both SK(p), SW(p) and their union ?
3. When, in general, is SK(p) = SW(p) ?

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ABOUT THE SMARANDACHE COMPLEMENTARY CUBIC FUNCTION

by **Marcela Popescu and Mariana Nicolescu**

DEFINITION. *Let* $g:\mathbf{N}^* \to \mathbf{N}^*$ *be a numerical function defined by* $g(n) = k$, *where* k *is the smallest natural number such that* nk *is a perfect cube:* $nk = s^3, s \in \mathbf{N}^*$.

Examples: 1) $g(7)=49$ because 49 is the smallest natural number such that $7 \cdot 49 = 7 \cdot 7^2 = 7^3$;

2) $g(12) = 18$ because 18 is the smallest natural number such that $12 \cdot 18 = (2^2 \cdot 3) \cdot (2 \cdot 3^2) = 2^3 \cdot 3^3 = (2 \cdot 3)^3$;

3) $g(27) = g(3^3) = 1$;

4) $g(54) = g(2 \cdot 3^3) = 2^2 = g(2)$.

PROPERTY 1. *For every* $n \in \mathbf{N}^*$, $g(n^3) = 1$ *and for every prime* p *we have* $g(p) = p^2$.

PROPERTY 2. *Let* n *be a composite natural number and* $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$, $0 < p_{i_1} < p_{i_2} < \cdots < p_{i_r}$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} \in \mathbf{N}^*$ *its prime factorization. Then* $g(n) = p_{i_1}^{d(\overline{\alpha_{i_1}})} \cdot p_{i_2}^{d(\overline{\alpha_{i_2}})} \cdot \ldots \cdot p_{i_r}^{d(\overline{\alpha_{i_r}})}$, *where* $\overline{\alpha_{i_j}}$ *is the remainder of the division of* α_{i_j} *by 3 and* $d:\{0,1,2\} \to \{0,1,2\}$ *is the numerical function defined by* $d(0) = 0, d(1) = 2$ *and* $d(2) = 1$.

If we take into account of the above definition of the function g, it is easy to prove the above properties.

OBSERVATION: $d(\overline{\alpha_{i_j}}) = \overline{\overline{3 - \overline{\alpha_{i_j}}}}$, for every $\alpha_{i_j} \in \mathbf{N}^*$, and in the sequel we use this writing for its simplicity.

REMARK 1. *Let* $m \in \mathbf{N}^*$ *be a fixed natural number. If we consider now the numerical function* $\tilde{g}:\mathbf{N}^* \to \mathbf{N}^*$ *defined by* $\tilde{g}(n) = k$, *where* k *is the smallest natural number such that* $nk = s^m, s \in \mathbf{N}^*$, *then we can observe that* $\tilde{g}$ *generalize the function* g, *and we also have:* $\tilde{g}(n^m) = 1$, $\forall n \in \mathbf{N}^*$, $\tilde{g}(p) = p^{m-1}$, $\forall p$ *prime and* $\tilde{g}(n) = p_{i_1}^{\overline{\overline{m - \alpha_{i_1}}}} \cdot p_{i_2}^{\overline{\overline{m - \alpha_{i_2}}}} \cdot \ldots \cdot p_{i_r}^{\overline{\overline{m - \alpha_{i_r}}}}$, *where* $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ *is the prime factorization of* n *and* $\overline{\alpha_{i_j}}$ *is the remainder of the division of* α_{i_j} *by* m, *therefore the both above properties holds for* $\tilde{g}$, *too.*

REMARK 2. *Because* $1 \le g(n) \le n^2$, *for every* $n \in \mathbf{N}^*$, *we have:* $\frac{1}{n} \le \frac{g(n)}{n} \le n$, *thus* $\sum_{n \ge 1} \frac{g(n)}{n}$ *is a divergent serie.*

In a similar way, using that we have $1 \le \tilde{g}(n) \le n^{m-1}$ *for every* $n \in \mathbf{N}^*$, *it results that* $\sum_{n\ge 1} \frac{\tilde{g}(n)}{n}$ *is also divergent.*

PROPERTY 3. *The function* $g:\mathbf{N}^* \to \mathbf{N}^*$ *is multiplicatve:* $g(x\cdot y) = g(x)\cdot g(y)$ *for every* $x,y \in \mathbf{N}^*$ *with* $(x,y) = 1$.

Proof. For $x = 1 = y$ we have $(x,y) = 1$ and $g(1\cdot 1) = g(1)\cdot g(1)$. Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ and $y = q_{j_1}^{\beta_{j_1}} \cdot q_{j_2}^{\beta_{j_2}} \cdot \ldots \cdot q_{j_s}^{\beta_{j_s}}$ be the prime factorization of x and y, repectively, so that $x\cdot y = 1$.

Because $(x,y) = 1$ we have $p_{i_h} \ne q_{j_k}$, for every $h = \overline{1,r}$ and $k = \overline{1,s}$.

Then $g(x\cdot y) = p_{i_1}^{\overline{3-\alpha_{i_1}}} \cdot p_{i_2}^{\overline{3-\alpha_{i_2}}} \cdot \ldots \cdot p_{i_r}^{\overline{3-\alpha_{i_r}}} \cdot q_{j_1}^{\overline{3-\beta_{j_1}}} \cdot q_{j_2}^{\overline{3-\beta_{j_2}}} \cdot \ldots \cdot q_{j_s}^{\overline{3-\beta_{j_s}}} = g(x)\cdot g(y)$.

REMARK 3. *The property holds also for the function* $\tilde{g}:\tilde{g}(x\cdot y) = \tilde{g}(x)\cdot\tilde{g}(y)$, *where* $(x,y) = 1$.

PROPERTY 4. *If* $(x,y) = 1$, x *and* y *are not perfect cubes and* $x,y>1$, then *the equation* $g(x) = g(y)$ *has not natural solutions.*

Proof. Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ and $y = \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}}$ (where $p_{i_h} \ne q_{j_k}, \forall h = \overline{1,r}, k = \overline{1,s}$, because $(x,y) = 1$) be their prime factorizations. Then $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}}$ and $g(y) = \prod_{k=1}^{s} q_{j_k}^{\overline{3-\beta_{j_k}}}$ and there exist at least $\overline{\alpha}_{i_h} \ne 0$ and $\overline{\beta}_{j_k} \ne 0$ (because x and y are not perfect cubes), therefore $1 \ne p_{i_h}^{\overline{3-\alpha_{i_h}}} \ne q_{j_k}^{\overline{3-\beta_{j_k}}} \ne 1$, so $g(x) \ne g(y)$.

CONSEQUENCE 1. *The equation* $g(x) = g(x+1)$ *has not natural solutions because for* $x \ge 1$, x *and* $x+1$ *are not both perfect cubes and* $(x, x+1) = 1$.

REMARK 4. *The property and the consequence is also true for the function* $\tilde{g}$: if $(x,y) = 1$, $x > 1$, $y > 1$, *and it does not exist* $a,b \in \mathbf{N}^*$ *so that* $x = a^m$, $y = b^m$ *(where m is fixed and has the above significance), then the equation* $\tilde{g}(x) = \tilde{g}(y)$ *has not natural solutions; the equation* $\tilde{g}(x) = \tilde{g}(x+1)$, $x \ge 1$ *has not natural solutions, too.*

It is easy to see that the proofs are similary, but in this case we denote by $\overline{\alpha}_{ij} = \alpha_{ij} \pmod m$ *and we replace* $\overline{3-\alpha_{i_1}}$ *by* $\overline{m-\alpha_{i_1}}$.

PROPERTY 5. *We have* $g(x\cdot y^3) = g(x)$, *for every* $x,y \in \mathbf{N}^*$.

Proof. If $(x,y) = 1$, then $(x,y^3) = 1$ and using property 1 and property 3, we have: $g(x\cdot y^3) = g(x)\cdot g(y^3) = g(x)$.

If $(x,y)=1$ we can write: $x=\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}$ where $p_{i_h} \neq d_{l_t}, q_{j_k} = d_{l_t}, p_{i_h} = q_{j_k}, \forall h=\overline{1,r}, k=\overline{1,s}$, $t=\overline{1,n}$. We have $g(x\cdot y^3)=g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{3\beta_{l_t}}) = g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}}) = g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}}) \cdot g(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+3\beta_{l_t}}) =$

$= \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-3\beta_{j_k}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\alpha_{l_t}-3\beta_{l_t}}} = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}} \cdot \prod_{t=1}^{r} d_{l_t}^{\overline{3-\alpha_{l_t}}} = g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}) \cdot g(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}) =$

$= g(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}) = g(x)$.

We used that $(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}) = 1$ and $(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{3\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}-3\beta_{l_t}}) = 1$ and the above properties.

REMARK 5. *It is easy to see that we also have* $\tilde{g}(x\cdot y^m)=\tilde{g}(x)$, *for every* $x,y\in\mathbf{N}^*$.

OBSERVATION . *If* $\frac{x}{y}=\frac{u^3}{v^3}$, *where* $\frac{u}{v}$ *is a simplified fraction, then g(x)=g(y). It is easy to prove this because* $x=ku^3$ *and* $y=kv^3$, *and using the above property we have:*

$$g(x)=g(k\cdot u^3)=g(k)=g(k\cdot v^3)=g(y)$$

OBSERVATION. If $\frac{x}{y}=\frac{u^m}{v^m}$ where $\frac{u}{v}$ is a simplified fraction, then, using remark 5, we have $\tilde{g}(x)=\tilde{g}(y)$, too.

CONSEQUENCE 2. *For every* $x\in\mathbf{N}^*$ *and* $n\in\mathbf{N}$,

$$g(x^n)=\begin{cases}1, \text{ if } n=3k;\\ g(x), \text{ if } n=3k+1;\\ \hat{g}(x), \text{ if } n=3k+2,\ k\in\mathbf{N},\end{cases}$$

where $\hat{g}(x)=g(g(x))$.

Proof. If n=3k, then x^n is a perfect cube, therefore $g(x^n)=1$.
If n=3k+1, then $g(x^n)=g(x^{3k}\cdot x)=g(x^{3k})\cdot g(x)=g(x)$.
If n=3k+2, then $g(x^n)=g(x^{3k}\cdot \hat{x})=g(x^{3k})\cdot g(\hat{x})=g(\hat{x})$.

PROPERTY 6. $g(x^2)=g^2(x)$, for every $x\in\mathbf{N}^*$.

Proof. Let $x=\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ be the prime factorization of x. Then $g(x^2)=g(\prod_{h=1}^{r} p_{i_h}^{2\alpha_{i_h}})=\prod_{h=1}^{r} p_{i_h}^{\overline{3-2\alpha_{i_h}}}$ and $g^2(x)=g(g(x))=g(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}})=\prod_{h=1}^{r} p_{i_h}^{\overline{3-3-\alpha_{i_h}}}$, but it is easy to observe that $\overline{3-2\alpha_{i_h}}=\overline{3-3-\alpha_{i_h}}$, because for :

$\overline{\alpha_{i_h}} = 0 \quad \overline{3-2\alpha_{i_h}} = \overline{3-0} = 0$ and $\overline{3-3-\alpha_{i_h}} = \overline{3-3-0} = \overline{3-0} = 0$

$\overline{\alpha_{i_h}} = 1 \quad \overline{3-2\alpha_{i_h}} = \overline{3-2} = 1$ and $\overline{3-3-\alpha_{i_h}} = \overline{3-3-1} = \overline{3-2} = 1$

$\overline{\alpha_{i_h}} = 2 \quad \overline{3-2\alpha_{i_h}} = \overline{3-4} = \overline{3-1} = 2$ and $\overline{3-3-\alpha_{i_h}} = \overline{3-3-2} = \overline{3-1} = 2,$

therefore $g(x^2) = g^2(x)$.

REMARK 6. *For the function* $\tilde{g}$ *is not true that* $\tilde{g}(x^2) = \tilde{g}^2(x)$, $\forall x \in \mathbf{N}^*$. *For example, for* $m=5$ *and* $x = 3^2$. $\tilde{g}(x^2) = \tilde{g}(3^4) = 3$ *while* $\tilde{g}(\tilde{g}(3^2)) = \tilde{g}(3^3) = 3^2$.

More generally $\tilde{g}(x^k) = \tilde{g}^k(x)$, $\forall x \in \mathbf{N}^*$ is not true. But for particular values of m.k and x the above equality is possible to be true. For example for $m = 6$, $x = 2^2$ and $k = 2$: $\tilde{g}(x^2) = \tilde{g}(2^4) = 2^2$ and $\tilde{g}^2(x) = \tilde{g}(\tilde{g}(2^2)) = \tilde{g}(2^4) = 2^2$.

REMARK 6'. *a)* $\tilde{g}(x^{m-1}) = \tilde{g}^{m-1}(x)$ *for every* $x \in \mathbf{N}^*$ *iff* m *is an odd number, because we have* $\overline{m-(m-1)\alpha_{i_h}} = \underbrace{\overline{m-m-\ldots-m-\alpha_{i_h}}}_{\text{m-1 times}}$, *for every* $\alpha_{i_h} \in \mathbf{N}$.

Example: *For* $m = 5$, $\tilde{g}(x^4) = \tilde{g}^4(x)$, *for every* $x \in \mathbf{N}^*$.

b) $\tilde{g}(x^{m-1}) = \tilde{g}^m(x)$. *for every* $x \in \mathbf{N}^*$ *iff* m *is an even number, because we have* $\overline{m-(m-1)\alpha_{i_h}} = \underbrace{\overline{m-m-\ldots-m-\alpha_{i_h}}}_{\text{m times}}$, *for every* $\alpha_{i_h} \in \mathbf{N}$.

Example: *For* $m = 4$. $\tilde{g}(x^3) = \tilde{g}^4(x)$. *for every* $x \in \mathbf{N}^*$.

PROPERTY 7. *For every* $x \in \mathbf{N}^*$ *we have* $g^3(x) = g(x)$.

Proof. Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}$ be the prime factorization of x. We saw that $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}}$ and

$g^3(x) = g(g^2(x)) = g(\prod_{h=1}^{r} p_{i_h}^{\overline{3-3-\alpha_{i_h}}}) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-3-3-\alpha_{i_h}}}$.

But $\overline{3-\alpha_{i_h}} = \overline{3-3-3-\alpha_{i_h}}$, for every $\alpha_{i_h} \in \mathbf{N}$, because for:

$\overline{\alpha_{i_h}} = 0 \quad \overline{3-\alpha_{i_h}} = 0$ and $\overline{3-3-3-\alpha_{i_h}} = \overline{3-3-3-0} = \overline{3-3-0} = \overline{3-0} = 0$

$\overline{\alpha_{i_h}} = 1 \quad \overline{3-\alpha_{i_h}} = 2$ and $\overline{3-3-3-\alpha_{i_h}} = \overline{3-3-3-1} = \overline{3-3-2} = \overline{3-1} = 2$

$\overline{\alpha_{i_h}} = 2 \quad \overline{3-\alpha_{i_h}} = 1$ and $\overline{3-3-3-\alpha_{i_h}} = \overline{3-3-3-2} = \overline{3-3-1} = \overline{3-2} = 1,$

therefore $g^3(x) = g(x)$, for every $x \in \mathbf{N}^*$.

REMARK 7. *For every* $x \in \mathbf{N}^*$ *we have* $\tilde{g}^3(x) = \tilde{g}(x)$ because $\overline{m-\overline{\alpha_{i_h}}} = \overline{m-m-m-\overline{\alpha_{i_h}}}$, for every $\alpha_{i_h} \in \mathbf{N}$. For $\overline{\alpha_{i_h}} = a \in \{1,\dots,m-1\} = A$, we have $\overline{m-\overline{\alpha_{i_h}}} = m-a \in A$, therefore $\overline{m-\overline{m-\overline{\alpha_{i_h}}}} = \overline{m-(m-a)} = \bar{a} = a$, so that $\overline{m-m-\overline{m-\overline{\alpha_{i_h}}}} = \overline{m-a} = \overline{m-\overline{\alpha_{i_h}}}$, which is also true for $\overline{\alpha_{i_h}} = 0$, therefore it is true for every $\alpha_{i_h} \in \mathbf{N}^*$.

PROPERTY 8. *For every* $x,y \in \mathbf{N}^*$ *we have* $g(x\cdot y) = g^2(g(x)\cdot g(y))$.

Proof. Let $x = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}$ and $y = \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}$ be the prime factorization of x and y, respectively, where $p_{i_h} \neq d_{l_t}, q_{j_k} \neq d_{l_t}, p_{i_h} \neq q_{j_k}, \forall h = \overline{1,r}, k = \overline{1,s}, t = \overline{1,n}$. Of course $x\cdot y = \prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}} \cdot \prod_{k=1}^{s} q_{j_k}^{\beta_{j_k}} \cdot \prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+\beta_{l_t}}$, so $g(x\cdot y) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\beta_{j_k}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-(\alpha_{l_t}+\beta_{l_t})}}$. On the other hand, $g(x) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\alpha_{l_t}}}$ and $g(y) = \prod_{k=1}^{s} q_{j_k}^{\overline{3-\beta_{j_k}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\beta_{l_t}}}$, so that

$$g^2(g(x)\cdot g(y)) = g^2\Big(\prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\beta_{j_k}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-\alpha_{l_t}}+\overline{3-\beta_{l_t}}}\Big) = \prod_{h=1}^{r} p_{i_h}^{\overline{3-3-3-\alpha_{i_h}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-3-3-\beta_{j_k}}} \cdot$$

$$\cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-3-(\overline{3-\alpha_{l_t}}+\overline{3-\beta_{l_t}})}} = \prod_{h=1}^{r} p_{i_h}^{\overline{3-\alpha_{i_h}}} \cdot \prod_{k=1}^{s} q_{j_k}^{\overline{3-\beta_{j_k}}} \cdot \prod_{t=1}^{n} d_{l_t}^{\overline{3-(\alpha_{l_t}+\beta_{l_t})}} = g(x\cdot y),$$

because $\overline{3-3-3-\bar{a}} = \overline{3-\bar{a}}$ and $\overline{3-3-(3-\bar{a}+3-\bar{b})} = \overline{3-(a+b)}, \forall a,b \in \mathbf{N}$.

REMARK 8. *In the case when* $(x,y) = 1$ *we obtain more simply the same result. Because* $(x,y) = 1 \Rightarrow (g(x),g(y)) = 1 \Rightarrow (g^2(x),g^2(y)) = 1$ *so we have:*

$$g^2(g(x)\cdot g(y)) = g(g(g(x)\cdot g(y))) = g(g(g(x))\cdot g(g(y))) = g(g^2(x)\cdot g^2(y)) =$$
$$= g(g^2(x))\cdot g(g^2(y)) = g^3(x)\cdot g^3(y) = g(x)\cdot g(y) = g(x\cdot y).$$

REMARK 9. *If* $(x,y) = 1$, *then* $g(xyz) = g^2(g(xy)\cdot g(z)) = g^2(g(x)g(y)g(z))$ *and this property can be extended for a finite number of factors, therefore if* $(x_1,x_2) = (x_2,x_3) = \dots = (x_{n-2},x_{n-1}) = 1$, *then* $g(\prod_{i=1}^{n} x_i) = g^2(\prod_{i=1}^{n} g(x_i))$.

PROPERTY 9. *The function g has not fixed points* $x \neq 1$.

Proof. We must prove that the equation $g(x) = x$ has not solutions $x>1$.

Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \dots \cdot p_{i_r}^{\alpha_{i_r}}, \alpha_{i_j} \geq 1, j = \overline{1,r}$ be the prime factorization of x. Then $g(x) = \prod_{j=1}^{r} p_{i_j}^{\overline{3-\alpha_{i_j}}}$ implies that $\alpha_{i_j} = \overline{3-\alpha_{i_j}}, \forall j \in \overline{1,r}$ which is not possible.

REMARK 10. *The function* $\tilde{g}$ *has fixed points only in the case* $m = 2k$, $k \in \mathbf{N}^*$. *These points are* $x = p_{i_1}^k \cdot p_{i_2}^k \cdot \dots \cdot p_{i_r}^k$, *where* $p_{i_j}, j = \overline{1,r}$ *are prime numbers.*

PROPERTY 10. *If* $\left(\frac{x}{(x,y)},y\right)=1$ *and* $\left(\frac{y}{(x,y)},x\right)=1$ *then we have* $g((x,y))=(g(x),g(y))$, *where we denote by (x,y) the greatest common divisor of x and y.*

Proof. Because $\left(\frac{x}{(x,y)},y\right)=1$ and $\left(\frac{y}{(x,y)},x\right)=1$, we have $\left(\frac{x}{(x,y)},(x,y)\right)=1$ *and* $\left(\frac{y}{(x,y)},(x,y)\right)=1$, then x and y have the following prime factorization: $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s}q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$, $p_{i_h}\neq d_{l_t}, q_{j_k}\neq d_{l_t}$, $p_{i_h}\neq q_{j_k}$, $\forall h=\overline{1,r}, k=\overline{1,s}, t=\overline{1,n}$. Then $(x,y)=\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$, therefore $g((x,y))=\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}}$. On the other hand

$$(g(x),g(y))=\left(\prod_{h=1}^{r}p_{i_h}^{\overline{3-\alpha_{i_h}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}},\prod_{k=1}^{s}q_{j_k}^{\overline{3-\beta_{j_k}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}}\right)=\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}}$$

and the assertion follows.

REMARK 11. *In the same conditions,* $\tilde{g}((x,y))=(\tilde{g}(x),\tilde{g}(y)), \forall x,y\in\mathbf{N}^*$.

PROPERTY 11. *If* $\left(\frac{x}{(x,y)},y\right)=1$ *and* $\left(\frac{y}{(x,y)},x\right)=1$ *then we have:* $g([x,y])=[g(x),g(y)]$, *where (x,y) has the above significance and* $[x,y]$ *is the least common multiple of x and y.*

Proof. We have the prime factorization of x and y used in the proof of the above property, therefore:

$$g([x\cdot y])=g\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{\beta_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}\right)=\prod_{h=1}^{r}p_{i_h}^{\overline{3-\alpha_{i_h}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{3-\beta_{j_k}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}}$$ and

$$[g(x),g(y)]=\left[\prod_{h=1}^{r}p_{i_h}^{\overline{3-\alpha_{i_h}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}},\prod_{k=1}^{s}q_{j_k}^{\overline{3-\beta_{j_k}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}}\right]=$$

$$=\prod_{h=1}^{r}p_{i_h}^{\overline{3-\alpha_{i_h}}}\cdot\prod_{k=1}^{s}q_{j_k}^{\overline{3-\beta_{j_k}}}\cdot\prod_{t=1}^{n}d_{l_t}^{\overline{3-\alpha_{l_t}}},$$

so we have $g([x,y])=[g(x),g(y)]$.

REMARK 12. *In the same conditions,* $\tilde{g}([x,y])=[\tilde{g}(x),\tilde{g}(y)], \forall x,y\in\mathbf{N}^*$.

CONSEQUENCE 4. *If* $\left(\frac{x}{(x,y)},y\right)=1$ *and* $\left(\frac{y}{(x,y)},x\right)=1$, *then* $g(x)\cdot g(y)=$ $=g((x,y))\cdot g([x,y])$ *for every* $x,y\in\mathbf{N}^*$.

Proof. Because $[x,y]=\frac{xy}{(x,y)}$ we have $[g(x),g(y)]=\frac{g(x)\cdot g(y)}{(g(x),g(y))}$ and using the last two properties we have:

$$g(x)\cdot g(y)=(g(x),g(y))\cdot[g(x),g(y)]=g((x,y))\cdot g([x,y]).$$

REMARK 13. *In the same conditions, we also have* $\tilde{g}(x)\cdot\tilde{g}(y)=\tilde{g}((x,y))\cdot\tilde{g}([x,y])$ *for every* $x,y\in\mathbf{N}^*$.

PROPERTY 13. *The sumatory numerical function of the function g is*

$$F(n)=\prod_{j=1}^{k}\left(\frac{\alpha_{i_j}+\overline{3-\alpha_{i_j}}}{3}(1+p_{i_j}+p_{i_j}^2)+h_{p_{i_j}}(\alpha_{i_j})\right),$$

where $n=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}}$ *is the prime factorization of n, and* $h_p:\mathbf{N}\to\mathbf{N}$ *is the numerical function defined by* $h_p(\alpha)=\begin{cases}1 \text{ for } \alpha=3k\\ -p \text{ for } \alpha=3k+1\\ 0 \text{ for } \alpha=3k+2\end{cases}$, *where p is a given number.*

Proof. Because the sumatory function of g is defined as $F(n)=\sum_{d/n}g(d)$ and because $(p_{i_1}^{\alpha_{i_1}},\prod_{t=2}^{k}p_{i_t}^{\alpha_{i_t}})=1$ and g is a multiplicative function, we have:

$F(n)=\left(\sum_{d_1/p_{i_1}^{\alpha_{i_1}}}g(d_1)\right)\cdot\left(\sum_{d_2/p_{i_2}^{\alpha_{i_2}}\ldots p_{i_k}^{\alpha_{i_k}}}g(d_2)\right)$ and so on, making a finite number of steps we obtain: $F(n)=\prod_{j=1}^{k}F(p_{i_j}^{\alpha_{i_j}})$.

But it is easy to prove that:

$$F(p^{\alpha})=\begin{cases}\frac{\alpha}{3}(1+p+p^2)+1 \text{ for } \alpha=3k;\\ \frac{\alpha+2}{3}(1+p+p^2)-p \text{ for } \alpha=3k+1;\\ \frac{\alpha+1}{3}(1+p+p^2) \text{ for } \alpha=3k,\ k\in\mathbf{N}, \text{ for every prime } p\end{cases}$$

Using the function h_p, we can write $F(p^{\alpha})=\frac{\overline{3-\alpha}}{3}(1+p+p^2)+h_p(\alpha)$, therefore we have the demanded expresion of F(n).

REMARK 14. *The expresion of F(n), where F is the sumatory function of* $\tilde{g}$, *is similary, but it is necessary to replace*

$\frac{\alpha_{i_j}+\overline{\overline{3-\alpha_{i_j}}}}{3}$ *by* $\frac{\alpha_{i_j}+\overline{\overline{m-\alpha_{i_j}}}}{m}$ *(where* $\overline{\alpha_{i_j}}$ *is now the remainder of the division of* α_{i_j} *by* m *and the sum* $1+p_{i_j}+p_{i_j}^2$ *by* $\sum_{k=0}^{m-1} p_{i_j}^k$*) and to define an adapted function* h_p.

In the sequel we study some equations which involve the function g.

1. *Find the solutions of the equations* $x\cdot g(x)=a$, *where* $x,a\in N^*$.

If a is not a perfect cube, then the above equation has not solutions.

If a is a perfect cube, $a=b^3, b\in N^*$, where $b=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}}$ is the prime factorization of b, then, taking into account of the definition of the function g, we have the solutions $x=b^3/d_{i_1i_2\ldots i_k}$ where $d_{i_1i_2\ldots i_k}$ can be every product $p_{i_1}^{\beta_1}p_{i_2}^{\beta_2}\cdots p_{i_k}^{\beta_k}$ where $\beta_1,\beta_2,\ldots,\beta_k$ take an arbitrary value which belongs of the set $\{0,1,2\}$.

In the case when $\beta_1=\beta_2=\cdots=\beta_k=0$ we find the special solution $x=b^3$, when $\beta_1=\beta_2=\cdots=\beta_k=1$, the solution $p_{i_1}^{3\beta_1-1}p_{i_2}^{3\beta_2-1}\cdots p_{i_k}^{3\beta_k-1}$ and when $\beta_1=\beta_2=\cdots=\beta_k=2$, the solution $p_{i_1}^{3\beta_1-2}p_{i_2}^{3\beta_2-2}\cdots p_{i_k}^{3\beta_k-2}$.

We find in this way $1+2C_k^1+2^2C_k^1+\cdots+2^kC_k^k=3^k$ different solutions, where k is the number of the prime divisors of b.

2. *Prove that the following equations have not natural solutions:*

$xg(x)+yg(y)+zg(z)=4$ *or* $xg(x)+yg(y)+zg(z)=5$. *Give a generalization.*

Because $xg(x)=a^3, yg(y)=b^3, zg(z)=c^3$ and the equations $a^3+b^3+c^3=4$ or $a^3+b^3+c^3=5$ have not natural solutions, then the assertion holds.

We can also say thet the equations $(xg(x))^n+(yg(y))^n+(zg(z))^n=4$ or $(xg(x))^n+(yg(y))^n+(zg(z))^n=5$ have not natural solutions, because the equations $a^{3n}+b^{3n}+c^{3n}=4$ or $a^{3n}+b^{3n}+c^{3n}=5$ have not.

3. *Find all solutions of the equation* $xg(x)-yg(y)=999$.

Because $xg(x)=a^3$ and $yg(y)=b^3$ we must give the solutions of the equation $a^3-b^3=999$, wh²ich are (a=10, b=1) and (a=12,b=9).

In the first case: a=10, b=1 we have $xa(x)=10^3=2^3\cdot5^3$

$$\Rightarrow x_0\in\left\{10^3,2^2\cdot5^3,2^3\cdot5^2,2\cdot5^3,2^3\cdot5,2^2\cdot5^2,2^2\cdot5,2\cdot5^2,2\cdot5\right\}$$

and $yb(y)=1\Rightarrow y_0=1$ so we have 9 different solutions (x_0,y_0).

In the second case: a=12, b=9 we have $xa(x)=12^3=2^6\cdot3^3$

$$\Rightarrow x_0\in\left\{2^6\cdot3^3,2^5\cdot3^3,2^6\cdot3^2,2^4\cdot3^3,2^6\cdot3,2^5\cdot3^2,2^4\cdot3^2,2^5\cdot3,2^4\cdot3\right\}$$

and $yb(y)=9^3=3^9\Rightarrow y_0\in\left\{3^9,3^8,3^7\right\}$ so we have another $9\cdot3=27$ different solutions (x_0,y_0).

4. *It is easy to observe that the equation g(x)=1 has an infinite number of solutions: all perfect cube numbers.*

5. *Find the solutions of the of the equation* $g(x)+g(y)+g(z)=g(x)g(y)g(z)$.

The same problem when the function is $\tilde{g}$.

It is easy to prove that the solutions are, in the first case, the permutations of the sets $\{u^3, 4v^3, 9t^3\}$, where $u,v,t \in N^*$, and in the second case $\{u^m, 2^{m-1}v^m, 3^{m-1}t^m\}$, $u,v,t \in N^*$.

Using the same ideea of [1], it is easy to find the solutions of the following equations which involve the function g:

a) $g(x)=kg(y)$, $k \in N^*, k>1$

b) $Ag(x)+Bg(y)+Cg(z)=0$, $A,B,C \in Z^*$

c) $Ag(x)+Bg(y)=C$, $A,B,C \in Z^*$, and to find also the solutions of the above equations when we replace the function g by $\tilde{g}$.

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Some Considerations Concerning the Sumatory Function Associated to Smarandache Function

by
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The Smarandache Function [4] ia a numerical function $S:N^* \to N^*$ defined by $S(n) = \min\{m | m! \text{ divisible by } n\}$.

From the definition it results that if

$$n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t} \tag{1}$$

is the decomposition of n into primes, then

$$S(n) = \max\{S(p_i^{r_i}) | i = 1,2,\dots,t\} \tag{2}$$

It is said that for every function f it can be attashed the sumatory function

$$F(n) = \sum_{d|n} f(d) \tag{3}$$

If f is the Smarandache function and $n = p^a$, then

$$F_s(p^a) = \sum_{j=0}^{a} S(p^j) = \sum_{j=0}^{a} S_p(j) \tag{4}$$

In [2] it is proved that

$$S(p^j) = (p-1)j + \sigma_{[p]}(j) \tag{5}$$

where

$$j = \sum_{i=1}^{t_j} k_i^j a_i(p) \tag{6}$$

and

$$\sigma_{[p]}(j) = \sum_{i=1}^{t_j} k_i^j \tag{7}$$

is the sum of the digits of the integer j, written in the generalised scale

$[p]$: $a_1(p), a_2(p), \dots, a_k(p), \dots$

with

$$a_n(p) = \frac{p^n - 1}{p-1} \quad , n = 1,2,\dots$$

For example

$$p = p \cdot a_1(p);$$
$$p^p = (p-1) \cdot a_p(p) + 1 \cdot a_1(p);$$
$$p^t = (p-1) \cdot a_t(p) + 1;$$

and

$$\sigma_{[p]}(p^p) = p;$$
$$S(p^p) = p^2; \quad S(p^{p^p}) = (p-1)p^p + p.$$

In [3] it is proved that

$$F_s(p^a) = (p-1)\frac{a(a+1)}{2} + \sum_{j=1}^{a} \sigma_{[p]}(j) \tag{8}$$

In the following we give an algorithm to calculate the sumatory function, associated to the Smarandache function:

1. Calculating the generalised scale $[p]$ $: a_1(p), a_2(p), ..., a_n(p), ...$
2. Calculating the expresion of a in the scale $[p]$. Let $a_{[p]} = \overline{k_s k_{s-1} ... k_1}$.
3. For $i = 1,2,..., s$

   3.1. If $k_i \neq 0$

   then

   3.1.1. $v_i = a - a_i(p) + 1$

   3.1.2. $z_i = \left(\overline{k_s k_{s-1} ... k_{i+1}}\right)_{u=a_i(p)}$

   3.1.3. $h_i = v_i - z_i$

   else

   3.1.4. $b = \overline{k_s k_{s-1} ... k_{i+1} - 1 p 00...0}$

   3.1.5. $v_i = b - a_i(p) + 1$

   3.1.6. $z_i = \left(\overline{k_s k_{s-1} ... k_i}\right)_{u=a_i(p)}$

   3.1.7. $h_i = v_i - z_i$

   3.2. $A_i = \left[\frac{h_i}{a_{i+1}(p) - a_i(p)}\right]$

   3.3. $r_i = h_i - A_i\left(a_{i+1}(p) - a_i(p)\right)$

   3.4. $B_i = \left[\frac{r_i}{a_i(p)}\right]$

   3.5. $q_i = r_i - B_i * a_i(p)$

   3.6. $S_i = A_i a_i(p) \frac{p(p-1)}{2} + A_i p + a_i(p) \frac{B_i\left(B_i + 1\right)}{2} + q_i\left(B_i + 1\right)$

4. Calculating $F_s(p^a) = (p-1)\frac{a(a+1)}{2} + \sum_{i=1}^{a} S_i$, $\left(\sum_{j=1}^{a} \sigma_{[p]}(j) = \sum_{i \geq 1} S_i(a)\right)$.

A Pascal program has been designed to the calculus of $F_s(p^a)$:

```
uses dos,crt;
        type tablou=array[1..100] of real;
        var a,k,amare,bmare,niu,z,alfaa,betaa,ro,r,s:tablou;
alfa,p,ik,amax,beta,suma,fsuma,u:real;
i,dim,max,j:longint;
hour,min,sec,sec100:word;
{*****************************************************}
{Calc. scale p right}
procedure bazapd(var b:tablou;var p:real;var a:real; var dim:longint);
var i:longint;
begin
   for i:=1 to 100 do
   bi]:=0;
   b[1]:=1;
   i:=0;
```

```
      repeat
        i:=i+1;
          b[i]:=b[i-1]*p+1;
      until b[i]>a;
   dim:=i;
end;

{*********************************************************}
{write  alfa in the scale p right}
procedure nrbazapd(var a:tablou; var p:real;
var alfa:real; var k:tablou; var max:longint);
var m,i:longint;
d,r,prod:real;
begin
for i:=1 to 100 do
    k[i]:=0;
d:=alfa;
max:=trunc(ln((p-1)*d+1)/ln(p));
repeat
 m:=trunc(ln((p-1)*d+1)/ln(p));
  k[m]:=trunc(d/a[m]);
  r:=d-a[m]*k[m];
  d:=r ;
until r<p;
if r<>0 then
 k[1]:=r;
end;
{*********************************************}
{calc. z for given i }

procedure calcz(var k:tablou;var a:tablou;
var i:longint;var u:real; var z:tablou; var p:real);
var j,i1,ind:longint;
prod:real;
begin
 z[i]:=0;
    ind:=1;
 for j:=i+1 to max do
  begin
    if k[j]<>0 then
     begin
       prod:=1;
       if ind>1 then
          begin
          for i1:=1 to ind-1 do
            prod:=prod*p; {****}
          prod:=prod*u+a[ind-1];
         end
       else
         prod:=u;
       z[i]:=z[i]+k[j]*prod;
```

```
    end;
   ind:=ind+1;
  end;
end;
{*************************************************}
begin
clrscr;
write('       give p=');
readln(p);
write('       give alfa=');
readln(alfa);
gettime(hour,min,sec,sec100);
writeln('       Timp Start:',hour,':',min,':',sec,':',sec100);
bazapd(a,p,alfa,dim);
nrbazapd(a,p,alfa,k,max);
for i:=1 to max do
 begin
  if k[i]<>0 then
   begin
    niu[i]:=alfa-a[i]+1;
    u:=a[i];
    calcz(k,a,i,u,z,p);
    alfaa[i]:=niu[i]-z[i];
   end
  else
   begin
    for j:=1 to max do
    betaa[j]:=k[j];
    betaa[i]:=p;
    betaa[i+1]:=betaa[i+1]-1;
    for j:=1 to i-1 do
     betaa[j]:=0;
      {Write beta in the scale 10}
      beta:=0;
       for j:=1 to max do
    beta:=beta+betaa[j]*a[j];
    niu[i]:=beta-a[i]+1;
    u:=a[i];
    calcz(betaa,a,i,u,z,p);
    alfaa[i]:=niu[i]-z[i];
    end;
    amare[i]:=int(alfaa[i]/(a[i+1]-a[i]));
    r[i]:=alfaa[i]-amare[i]*(a[i+1]-a[i]);
    bmare[i]:=int(r[i]/a[i]);
    ro[i]:=r[i]-bmare[i]*a[i];
    s[i]:=amare[i]*a[i]*(p*(p-1)/2)+amare[i]*p;
    s[i]:=s[i]+a[i]*(bmare[i]*(bmare[i]+1)/2);
    s[i]:=s[i]+ro[i]*(bmare[i]+1);
   end;
suma:=0;
```

```
for i:=1 to max do
 suma:=suma+s[i];
fsuma:=(p-1)*((alfa*(alfa+1))/2)+suma;
writeln('       fsuma=',fsuma);
gettime(hour,min,sec,sec100);
writeln('       Timp Stop:',hour,':',min,':',sec,':',sec100);
end.
```

We applied the algoritm for $p=3$ and $a = 300$ we obtain

TIMES START: 10:34:1:56

TIMES STOP: 10:34:1:57

We applied the formulas [4] for $p=3$ and $a = 300$ we obtain

TIMES START : 10:33:31:2

TIMES STOP: 10:33:31:95

A consequence of this work is that the proposed algoritm is faster then formula [4] .

From the Legendre formula it results that [1]

$$S_p(j)=p\left(j-i_p(j)\right) \text{ with } 0\le i_p(j)\le\left[\frac{j-1}{p}\right],$$

and

$$F_s\left(p^a\right)=\sum_{j=0}^{a}p\left(j-i_p(j)\right)=p\sum_{j=0}^{a}j-p\sum_{j=0}^{a}i_p\left(j\right),$$

consequently

$$F_s(p^a)=\frac{pa(a+1)}{2}-p\sum_{i=0}^{a}i_p(j) \tag{9}$$

In [1] it is showed that

$$i_p(j)=\frac{j-\sigma_{[p]}(j)}{p}.$$

In particular,

$$i_p(p)=0\,;\quad i_p(p^t)=p^{t-1}-1$$

and

$$\sum_{j=0}^{a}i_p(j)=\frac{1}{p}\left[\sum_{j=0}^{a}j-\sum_{i=1}^{i_j}k_i^j\right].$$

If $p\ge a$, then

$$j=j\cdot a_1(p),\quad \sigma_{[p]}(j)=j,\quad (j=1,2,...,a),\quad S(p^a)=pa$$

and

$$\sum_{j=0}^{a}\sigma_{[p]}(j)=\frac{a(a+1)}{2};\quad \sum_{j=0}^{a}i_p(j)=0$$

$$F_s(p^a)=\frac{pa(a+1)}{2} \tag{10}$$

For example

$$F_s(11^3)=s(1)+S(11)+S(11^2)+S(11^3)=66 \text{ or}$$

$$F_s(11^3)=\frac{11\cdot3\cdot4}{2}=66\,.$$

In particular,

$$F_s(p^p)=\frac{p^2(p+1)}{2} \tag{11}$$

If $p \le a$, then $a = pQ + R$ with $0 \le R \le p$, and

$$\sum_{j=0}^{a} i_p(j)=\sum_{j=0}^{a}\left\{\left[\frac{i}{p}\right]-\left[\frac{\sigma_p(j)}{p}\right]\right\} \Rightarrow$$

$$\sum_{j=0}^{a} i_p(j)=\frac{pQ(Q-1)}{2}+Q(R+1)-\sum_{j=p}^{a}\left[\frac{\sigma_p(j)}{p}\right],$$

consequently,

$$F_s(p^a)=\frac{pa(a+1)}{2}-\frac{p^2Q(Q-1)}{2}-pQ(R+1)+p\sum_{j=0}^{a}\left[\frac{\sigma_{[p]}(p)}{p}\right] \tag{12}$$

In particular, for $p = a$ then $Q = 1$, $R = 0$ and

$$F_s(p^p)=\frac{p^2(p+1}{2} \tag{13}$$

For example,

$$F_s(3^3)=18; \quad F_s(5^5)=75;$$

$$F_s(n^n)=\frac{2(2^p+1)^2(2^{p-2}+1)}{27}, \text{ for } n=\frac{2^p+1}{3}, \text{ with } 3<p\le 31 \text{ and } p \text{ prime.}$$

If $n = p^a q^b$ with $p < q$ and $p^a < q$, then

$$F_s(p^aq^b)=\sum_{d/p^aq^b} S(d)=\sum_{i=0}^{a}\sum_{j=0}^{a} S(p^iq^j)=(a+1)\sum_{j=0}^{b} S(q^j)=(a+1)F_s(q^b)$$

Then :

I. If $q \ge b$,

$$F_s(p^aq^b)=\frac{qb(a+1)(b+1)}{2} \tag{14}$$

II. If $q \le b$,

$$F_s(p^aq^b)=\frac{(a+1)qb(b+1)}{2}-\frac{(a+1)q^2\overline{Q}(\overline{Q}-1)}{2}-q(a+1)\overline{Q}(\overline{R}+1)+$$
$$q(a+1)\sum_{j=q}^{b}\left[\frac{\sigma_{[q]}(j)}{q}\right] \tag{15}$$

where $b = q\overline{Q} + \overline{R}$, with $0 \le \overline{R} \le q$.

If $n = p^a q$, then

$$F_s(p^aq)=\sum_{i=0}^{a} S(p^i)+\sum_{i=0}^{a} S(qp^i) \ .$$

For $p > q$, then $p^i > q$ and $S(qp^i) = S(p^i)$ with $i \ge 1$ consequently,

$$F_s(p^aq)=2F_s(p^a)+S(q)-1 \tag{16}$$

For $p < q$, there exists $x < a$ with $p^{x-1} < q < p^x$ and

$$S(qp^i)=\begin{cases} S(q), & i=0,1,...,x-1 \\ S(p^i), & i=x,.....,a \end{cases}$$

consequently,

$$F_s(p^aq)=\sum_{i=0}^{x-1} S(p^i)+xS(q)+2\sum_{i=x}^{a} S(p^i)$$

or

$$F_s(p^a q) = F_s(p^{x-1}) + xS(q) + (a+x)(a-x+1)(p-1) + 2\sum_{j=x}^{a} \sigma_{[p]}(j) \qquad (17)$$

For example, if $p \geq a$, then

$$F_S(p^a q) = F_s(p^{x-1}) + xS(q) + p(x+a)(a-x+1)$$

If $n = p^a q^k$ with $p > q$, then

$$F_s(p^a q^2) = \sum_{i=0}^{a} S(p^i) + \sum_{i=0}^{a} S(qp^i) + \sum_{i=0}^{a} S(q^2 p^i).$$

But $S(q^k p^i) = S(p^i)$ for $i \geq k$, because $\max\left(S(p^i), S(q^k)\right) = S(p^i)$ for $i \geq k$ consequently,

$$F_s(p^a q^2) = F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1 =$$
$$= 3F_s(p^a) + S(q) + S(q^2) + S(q^2 p) - p - 2$$

In short

$$F_s(p^a q) = 2F_s(p^a) + S(q) - 1$$
$$F_s(p^a q^2) = F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1$$
$$F_s(p^a q^3) = F_s(p^a q^2) + F_s(p^a) + S(q^3) + S(q^3 p) + S(q^3 p^2) - p - 2p - 1$$
$$\cdots\cdots\cdots\cdots$$
$$F_s(p^a q^k) = F_s(p^a q^{k-1}) + F_s(p^a) + S(q^k) + S(q^k p) + S(q^k p^2) +$$
$$+ \ldots + S(q^k p^{k-1}) - p - 2p - \ldots - (k-1)p - 1$$

Hence

$$F_s(p^a q^k) = (k+1)F_s(p^a) + \sum_{i=1}^{k} s(q^i) + \sum_{i=2}^{k} S(q^i p) + \sum_{i=3}^{k} S(q^i p^2) +$$
$$+ \ldots + \sum_{i=k-1}^{k} S(q^i p^{k-2}) + S(q^k p^{k-1}) - k - \frac{pk(k^2-1)}{6} \qquad (18)$$

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SOME ELEMENTARY ALGEBRAIC CONSIDERATIONS INSPIRED BY SMARANDACHE'S FUNCTION (II)

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In this paper we continue the algebraic consideration begun in [2]. As it was sun, two of the proprieties of Smarandache's function are hold:

(1) S is a surjective function;

(2) $S([m,n]) = \max\{S(m), S(n)\}$, where $[m,n]$ is the smallest common multiple of m and n.

That is on $\aleph$ there are considered both of the divisibility order "$\preceq_d$" having the known properties and the total order with the usual order $\leq$ with all its properties. $\aleph$ has also the algebric usual operations "$+$" and "$\cdot$". For instance:

$$a \leq b \Longleftrightarrow (\exists)\, u \in \aleph \text{ so that } b = a + u.$$

Here we can stand out:

: the universal algebra $(\aleph^*, \Omega)$, the set of operations is $\Omega = \{\vee_d, \varphi_0\}$ where $\vee_d : (\aleph^*)^2 \to \aleph^*$ is given by $m \vee_d n = [m,n]$, and $\varphi_0 : (\aleph^*)^0 \to \aleph^*$ the null operation that fixes 1-unique particular element with the role of neutral element for "$\vee_d$"-that means $\varphi_0(\{\emptyset\}) = 1$ and $1 = e_{\vee_d}$;

: the universal algebra $(\aleph^*, \Omega')$, the set of operations is $\Omega' = \{\vee, \psi_0\}$ where $\vee : \aleph^2 \to \aleph$ is given by $x \vee y = \sup\{x,y\}$ and $\psi_0 : \aleph^0 \to \aleph$ a null operation with $\psi_0(\{\emptyset\}) = 0$ the unique particular element with the role of neutral element for $\vee$, so $0 = e_{\vee}$.

We observe that the universal algebras $(\aleph^*, \Omega)$ and $(\aleph^*, \Omega')$ are of the same type:

$$\begin{pmatrix} \vee_d & \varphi_0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \vee & \psi_0 \\ 2 & 0 \end{pmatrix}$$

and with the similarity (bijective) $\vee_d \Longleftrightarrow \vee$ and $\varphi_0 \Longleftrightarrow \psi_0$, Smarandache's function $S : \aleph^* \to \aleph$ is a morphism surjective between them

$$\begin{aligned} S(x \vee_d y) &= S(x) \vee S(y), \forall x, y \in \aleph^* \text{ from (2) and} \\ S(\varphi_0(\{\emptyset\})) &= \psi_0(\{\emptyset\}) \Longleftrightarrow S(1) = 0. \end{aligned}$$

Problem 3. If $S : \aleph^* \to \aleph$ is Smarandache's function defined as we know by

$$S(n) = m \Longleftrightarrow m = \min\{k : n \text{ divides } k!\}$$

and I is a some set, then there exists an unique $s : (\aleph^*)^I \to \aleph^I$ a surjective morphisme between the universal algebras $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega'\right)$ so that $p_i \circ s = \varsigma \circ \tilde{p}_i$, for $i \in I$, where $p_j : \aleph^I \to \aleph$ defined by $a = \{a_i\}_{i \in I} \in \aleph^I$, $p_j(a) = a_j$, for each $j \in I$, p_j are the canonical projections, morphismes between $\left(\aleph^I, \Omega'\right)$ and $(\aleph, \Omega')$-universal algebras of the same kind and $\tilde{p}_j : (\aleph^*)^I \to \aleph^*$ analogously between $\left((\aleph^*)^I, \Omega\right)$ and $(\aleph^*, \Omega)$. We shall go over the following three steps in order to justify the assumption:

Theorem 0.1. *Let by $(\aleph, \Omega)$ is an universal algebra more complexe with*

$$\Omega = \{\vee_d, \wedge_d, \varphi_0, \overline{\varphi}_0\}$$

of the kind $\tau : \Omega \to \aleph$ given by

$$\tau = \begin{pmatrix} \vee_d & \wedge_d & \varphi_0 & \overline{\varphi}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$$

where $\vee_d$ and φ_0 are defined as above and $\wedge_d : \aleph^2 \to \aleph$, for each $x, y \in \aleph, x \wedge_d y = (x, y)$ where (x, y) is the biggest common divisor of x and y and $\overline{\varphi}_0 : \aleph^0 \to \aleph$ is the null operation that fixes 0-an unique particular element having the role of the neutral element for "$\wedge_d$" i.e. $\overline{\varphi}_0(\{\emptyset\}) = 0$ so $0 = e_{\wedge_d}$ and I a set. Then $\left(\aleph', \tilde{\Omega}\right)$ with $\tilde{\Omega} = \{\omega_1, \omega_2, \omega_0, \overline{\omega}_0\}$ becomes an universal algebra of the same kind as $(\aleph, \Omega)$ and the canonical projections become surjective morphismes between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(\aleph, \Omega)$, an universal algebra that satisfies the following property of universality:
$(\mathcal{U})$: for every $\left(A, \overline{\Omega}\right)$ with $\overline{\Omega} = \{\top, \bot, \sigma_0, \overline{\sigma}_0\}$ an universal algebra of the same kind

$$\tau = \begin{pmatrix} \top & \bot & \sigma_0 & \overline{\sigma}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$$

and $u_i : A \to \aleph$, for each $i \in I$, morphismes between $\left(A, \overline{\Omega}\right)$ and $(\aleph, \Omega)$, exists an unique $u : A \to \aleph^I$ morphism between the universal algebras $\left(A, \overline{\Omega}\right)$ and $\left(\aleph^I, \tilde{\Omega}\right)$ so that $p_j \circ u = u_j$, for each $j \in I$, where $p_j : \aleph^I \to \aleph$ with each $a = \{a_i\}_{i \in I} \in \aleph^I, p_j(a) = a_j$, for each $j \in I$ are the canonical projections morphismes between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(\aleph, \Omega)$.

Proof. Indeed $\left(\aleph^I, \tilde{\Omega}\right)$ with $\tilde{\Omega} = \{\omega_1, \omega_2, \omega_0, \overline{\omega}_0\}$ becomes an universal algebra because we can well define:

ω_1 : $(\aleph^I)^2 \to \aleph^I$ by each $a = \{a_i\}_{i\in I}, b = \{b_i\}_{i\in I} \in \aleph; \omega_1(a,b) = \{a_i \vee_d b_i\}_{i\in I} \in \aleph^I$ and

ω_2 : $(\aleph^I)^2 \to \aleph^I$ by $\omega_2(a,b) = \{a_i \wedge_d b_i\}_{i\in I} \aleph^I$ and also

ω_0 : $(\aleph^I)^0 \to \aleph^I$ with $\omega_0(\{\emptyset\}) = \{e_i = 1\}_{i\in I} \in \aleph^I$

an unique particular element (the family with all the components equal with 1) fixed by ω_0 and having the role of neutral for the operation ω_1 noted with e_{ω_1} and then $\overline{\omega}_0 : (\aleph^I)^0 \to \aleph^I$ with $\overline{\omega}_0(\{\emptyset\}) = \{\overline{e}_i = 0\}_{i\in I}$ an unique particular element fixed by $\overline{\omega}_0$ but hawing the role of neutral for the operation ω_2 noted $\overline{e}_{\omega_2}$ (the verifies are imediate).

The canonical projections $p_j : \aleph^I \to \aleph$, defined as above, become morphismes between $\left(\aleph^I, \tilde{\Omega}\right)$ and $(\aleph, \Omega)$. Indeed the two universal algebras are of the same kind

$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_0 & \overline{\omega}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \vee_d & \wedge_d & \varphi_0 & \overline{\varphi}_0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$$

and with the similairity (bijective) $\omega_1 \Longleftrightarrow \vee_d; \omega_2 \Longleftrightarrow \wedge_d; \omega_0 \Longleftrightarrow \varphi_0; \overline{\omega}_0 \Longleftrightarrow \overline{\varphi}_0$ we observe first that for each $a, b \in \aleph^I, p_j(\omega_1(a,b)) = p_j(a) \vee_d p_j(b)$, for each $j \in I$ because $a = \{a_i\}_{i\in I}, b = \{b_i\}_{i\in I}, p_j(\omega_1(a,b)) = p_j\left(\{a_i \vee_d b_i\}_{i\in I}\right) = a_j \vee_d b_j$ and $p_j(a) \vee_d p_j(b) = p_j(\{a_i\}_{i\in I}) \vee_d p_j\left(\{b_i\}_{i\in I}\right) = a_j \vee_d b_j$ and then $p_j(\omega_0(\{\emptyset\})) = \varphi_0(\{\emptyset\}) \Longleftrightarrow p_j\left(\{e_i = 1\}_{i\in I}\right) = 1 \Longleftrightarrow p_j(e_{\omega_1}) = e_{\vee_d}$; analogously we prove that p_j, for each $j \in I$ keeps the operations ω_2 and $\overline{\omega}_0$, too. So, it was built the universal algebra $\left(\aleph^I, \tilde{\Omega}\right)$ with $\tilde{\Omega} = \{\omega_1, \omega_2, \omega_0, \overline{\omega}_0\}$ of the kind τ described above.

We prove the property of universality $(\mathcal{U})$.

We observe for this purpose that the u_i morphismes for each $i \in I$, presumes the coditions: for each $x, y \in S, u_i(x \top y) = u_i(x) \vee_d u_i(y); u_i(x \perp y) = u_i(x) \wedge_d u_i(y); u_i(\sigma_0(\{\emptyset\})) = \varphi_0(\{\emptyset\}) \Longleftrightarrow u_i(e_\top) = e_{\vee_d} = 1$ and $u_i(\overline{\sigma}_0(\{\emptyset\})) = \overline{\varphi}_0(\{\emptyset\}) \Longleftrightarrow u_i(\overline{e}_\perp) = e_{\wedge_d} = 0$ which show also the similarity (bijective) between $\overline{\Omega}$ and Ω. We also observe that $(S, \overline{\Omega})$ and $\left(\aleph^I, \tilde{\Omega}\right)$ are of the same kind and there is a similarity (bijective) between $\overline{\Omega}$ and $\tilde{\Omega}$ given by $\top \Longleftrightarrow \omega_1; \perp \Longleftrightarrow \omega_2; \sigma_0 \Longleftrightarrow \omega_0; \overline{\sigma}_0 \Longleftrightarrow \overline{\omega}_0$.

We define the corespondance $u : A \to \aleph^I$ by $u(x) = \{u_i(x)\}_{i\in I}$.

u is the function:

- for each $x \in A, (\exists)\, u_i(x) \in \aleph$ for each $i \in I$ (u_i-functions) so $(\exists)\, \{u_i(x)\}_{i\in I}$ that can be imagines for x;

- $x_1 = x_2 \Longrightarrow u(x_1) = u(x_2)$ because $x_1 = x_2$ and u_i-functions lead to $u_i(x_1) = u_i(x_2)$ for each $i \in I \Longrightarrow \{u_i(x_1)\}_{i\in I} = \{u_i(x_2)\}_{i\in I} \Longrightarrow u(x_1) = u(x_2)$.

u is a morphisme: for each $x, y \in A, u(x \top y) = \{u_i(x \top y)\}_{i\in I} = \{u_i(x) \vee_d u_i(y)\}_{i\in I} = \omega_1\left(\{u_i(x)\}_{i\in I}, \{u_i(y)\}_{i\in I}\right) = \omega_1(u(x), u(y))$. Then $u(\sigma_0(\{\emptyset\})) = \omega_0(\{\emptyset\}) \Longleftrightarrow u(e_\top) = e_{\omega_1}$ because for each $\{a_i\}_{i\in I} \in \aleph^I, \omega_1(\{a_i\}_{i\in I}, \{u_i(e_\top)\}_{i\in I}) = \{a_i \vee_d u_i(e_\top)\}_{i\in I} = \{a_i \vee_d 1\}_{i\in I} = \{a_i\}_{i\in I}$.

Analogously we prove that u keeps the operations: $\perp$ and $\overline{\sigma}_0$.

Besides the condition $p_j \circ u = u_j$, for each $j \in I$ is verified (by the definition: for each $x \in S$, $(p_j \circ u)(x) = p_j(u(x)) = p_j(\{u_i(x)\}_{i\in I}) = u_j(x)$).

For the singleness of u we consider u and $\overline{u}$, two morphismes so that $p_j \circ u = u_j$ (1) and $p_j \circ \overline{u} = u_j$ (2), for every $j \in I$. Then for every $x \in A$, if $u(x) = \{u_i(x)\}_{i\in I}$ and $\overline{u}(x) = \{z_i\}_{i\in I}$ we can see that $y_j = u_j(x) = (p_j \circ \overline{u})(x) = p_j(\{z_i\}_{i\in I}) = z_j$, for every $j \in I$ i.e. $u(x) = \overline{u}(x)$, for every $x \in A \Longleftrightarrow u = \overline{u}$.

Consequence. Particularly, taking $A = \aleph^I$ and $u_i = p_i$ we obtain: the morphisme $u : \aleph^I \to \aleph^i$ verifies the condition $p_j \circ u = p_j$, for every $j \in I$, if and only if, $u = 1_{\aleph^I}$.

The property of universality establishes the universal algebra $\left(\aleph^I, \tilde{\Omega}\right)$ until an isomorphisme as it results from:

Theorem 0.2. *If (P, Ω) is an universal algebra of the same kind as $(\aleph, \Omega)$ and $p'_i : P \to \aleph$, $i \in I$ a family of morphismes between (P, Ω) and $(\aleph, \Omega)$ so that for every universal algebra $\left(A, \overline{\Omega}\right)$ and every morphisme $u_i : A \to \aleph$, for every $i \in I$ between $\left(A, \overline{\Omega}\right)$ and $(\aleph, \Omega)$ it exists an unique morphisme $u : A \to P$ with $p'_j \circ u = u_i$, for every $i \in I$, then it exists an unique isomorphisme $f : P \to \aleph^I$ with $p_i \circ f = p'_i$, for every $i \in I$.*

Proof. From the property of universality of $\left(\aleph^I, \tilde{\Omega}\right)$ it results an unique $f : P \to \aleph^I$ so that for every $i \in I, p_i \circ f = p'_i$ with f morphisme between (P, Ω) and $\left(\aleph^I, \tilde{\Omega}\right)$. Applying now the same property of universality to $(P, \Omega) \Longrightarrow$exists an unique $\overline{f} : \aleph^I \to P$ so that $p'_i \circ \overline{f} = p_i$, for every $i \in I$ with $\overline{f}$ morphisme between $\left(\aleph^I, \tilde{\Omega}\right)$ and (P, Ω). Then $p'_j \circ \overline{f} = p_j \Longleftrightarrow p_j \circ \left(f \circ \overline{f}\right) = p_j$, using the last consequence, we get $f \circ \overline{f} = 1_{\aleph^I}$. Analogously, we prove that $f \circ \overline{f} = 1_P$ from where $\overline{f} = f^{-1}$ and the morphisme f becomes isomorphisme.

We could emphasize other properties (a family of finite support or the case $I-$filter) but we remain at these which are strictly necessary to prove the proposed assertion (Problem 3).

b) Firstly it was built $\left(\aleph^I, \tilde{\Omega}\right)$ being an universal algebra more complexe (with four operations). We try now a similar construction starting from $(\aleph, \Omega^*)$ with $\Omega^* =$

$(\vee, \wedge, \psi_0)$ with "$\vee$" and "ψ_0" defined as above and $\wedge : \aleph^2 \to \aleph$ with $x \wedge y = \inf\{x, y\}$ for every $x, y \in \aleph$. ■

Theorem 0.3. *Let by $(\aleph, \Omega^*)$ the above universal algebra and I a set. Then:*
(i) $\left(\aleph^I, \theta\right)$ with $\theta = \{\theta_1, \theta_2, \theta_0\}$ becomes an universal algebra of the same kind τ as $(\aleph, \Omega^*)$ so $\tau : \theta \to \aleph$ is

$$\tau = \begin{pmatrix} \theta_1 & \theta_2 & \theta_0 \\ 2 & 2 & 0 \end{pmatrix};$$

(ii) For every $j \in I$ the canonical projection $p_j : \aleph^I \to \aleph$ defined by every $a = \{a_i\}_{i \in I} \in \aleph^I, p_j(a) = a_j$ is a surjective morphisme between $\left(\aleph^I, \theta\right)$ and $(\aleph, \Omega^*)$ and $\ker p_j = \left\{a \in \aleph^I : a = \{a_i\}_{i \in I} \text{ and } a_j = 0\right\}$ where by definition we have $\ker p_j = \left\{a \in \aleph^I : p_j(a) = e_\vee\right\}$;
(iii) For every $j \in I$ the canonical injection $q_j : \aleph \to \aleph^I$ for every $x \in \aleph, q_j(x) = \{a_i\}_{i \in I}$ where $a_i = 0$ if $i \neq j$ and $a_j = x$ is an injective morphisme between $(\aleph, \Omega^*)$ and $\left(\aleph^I, \theta\right)$ and $q_j(\aleph) = \left\{\{a_i\}_{i \in I} : a_i = 0, \forall i \in I - \{j\}\right\}$;
(iv) If $j, k \in I$ then:

$$p_j \circ q_k = \begin{cases} \mathcal{O}\text{-the null morphisme} & \text{for } j \neq k, \\ 1_\aleph\text{-the identical morphisme} & \text{for } j = k. \end{cases}$$

Proof. *(i)* We well define the operations $\theta_1 : \left(\aleph^I\right)^2 \to \aleph^I$ by $\forall a = \{a_i\}_{i \in I} \in \aleph^I$ and $b = \{b_i\}_{i \in I} \in \aleph^I, \theta_1(a, b) = \{a_i \vee b_i\}_{i \in I}; \theta_2 : \left(\aleph^I\right)^2 \to \aleph^I$ by $\theta_2(a, b) = \{a_i \wedge b_i\}_{i \in I}$ and $\theta_0 : \left(\aleph^I\right)^0 \to \aleph^I$ by $\theta_0(\{\emptyset\}) = \{e_i = 0\}_{i \in I}$ an unique particular element fixed by θ_0, but with the role of neutral element for θ_1 and noted e_{θ_1} (the verifications are immediate).

(ii) The canonical projections are proved to be morphismes (see the step a)), they keep all the operations and

$$\ker p_j = \left\{a = \{a_i\}_{i \in I} \in \aleph^I : p_j(a) = e_\vee\right\} = \left\{a \in \aleph^I : a_j = 0\right\}.$$

(iii) For every $x, y \in \aleph, q_j(x \vee y) = \{c_i\}_{i \in I}$ where $c_i = 0$ for every $i \neq j$ and $c_j = x \vee y$ and

$$\theta_1\left(\left\{\begin{array}{l} a_i = 0, \quad \forall i \neq j \\ a_j = x \end{array}\right\}, \left\{\begin{array}{l} b_i = 0, \quad \forall i \neq j \\ b_j = y \end{array}\right\}\right) = \left\{\begin{array}{ll} c_i = 0 & , \forall i \neq j \\ c_j = x \vee y & \end{array}\right\}$$

i.e. $q_j(x \vee y) = \theta_1(q_j(x), q_j(y))$ with $j \in I$, therefore q_j keeps the operation "$\vee$" for every $j \in I$. Then $q_j(\psi(\{\emptyset\}) = \theta_0(\{\emptyset\}) \Longleftrightarrow q_j(e_\vee) = \{e_i = 0\}_{i \in I} \Longleftrightarrow q_j(0) = \{e_i = 0\}_{i \in I} = e_{\theta_1}$ because $\forall a = \{a_i\}_{i \in I} \in \aleph^I, \theta_1(q_j(0), a) = \theta_1\left(\{e_i = 0\}_{i \in I}, \{a_i\}_{i \in I}\right) =$

$\{e_i \vee a_i\}_{i\in I} = \{a_i\}_{i\in I} = a$ enough for $q_j(0) = e_{\theta_1}$ because θ_1 is obviously comutative -this observation refers to all the similar situations met before. Analogously we also prove that θ_2 is kept by q_j and this one for every $j \in I$.

(*iv*) For every $x \in \aleph$, $(p_j \circ q_k)(x) = p_j(q_k(x)) = p_j\left(\left\{\begin{array}{ll} a_i = 0 & ,\forall i \neq k \\ a_k = x & \end{array}\right\}\right) =$ $0 \Longrightarrow p_j \circ q_k = \mathcal{O}$ for $j \neq k$ and $(p_j \circ q_j)(x) = p_j(q_j(x)) = p_j\left(\left\{\begin{array}{ll} a_i = 0 & ,\forall i \neq j \\ a_j = x & \end{array}\right\}\right) =$ $x \Longrightarrow p_j \circ q_k = 1_{\mathbf{N}}$ for $j = k$. ■

The universal algebra $\left(\aleph^I, \theta\right)$ satisfies the following property of universality:

Theorem 0.4. *For every* $\left(A, \bar{\theta}\right)$ *with* $\bar{\theta} = \{\top, \bot, \theta_0\}$ *an universal algebra of the some kind* $\tau : \bar{\theta} \to \aleph$

$$\tau = \begin{pmatrix} \top & \bot & \theta_0 \\ 2 & 2 & 0 \end{pmatrix}$$

as $\left(\aleph^I, \theta\right)$ *and* $u_i : A \to \aleph$ *for every* $i \in I$ *morphismes between* $\left(A, \bar{\theta}\right)$ *and* $(\aleph, \Omega^*)$, *exists an unique* $u : A \to \aleph^I$ *morphisme between the universal algebras* $\left(A, \bar{\theta}\right)$ *and* $\left(\aleph^I, \theta\right)$ *so that* $p_j \circ u = u_j$, *for every* $j \in I$ *with* $p_j : \aleph^I \to \aleph, \forall \mathbf{a} = \{a_i\}_{i\in I} \in \aleph^I, p_j(a) = a_j$ *the canonical projections morphismes between* $\left(\aleph^I, \theta\right)$ *and* $(\aleph, \Omega^*)$.

Proof. The proof repeats the other one from the Theorem 1, step a). ■

The property of universality establishes the universal algebra $\left(\aleph^I, \theta\right)$ until an isomorphisme, which we can state by:

If (P, Ω^*) it is an universal algebra of the same kind as $(\aleph, \Omega^*)$ and $p_i' : P \to \aleph$ for every $i \in I$ a family of morphismes between (P, Ω^*) and $(\aleph, \Omega^*)$ so that for every universal algebra $\left(A, \bar{\theta}\right)$ and every morphismes $u_i : A \to \aleph, \forall i \in I$ between $\left(A, \bar{\theta}\right)$ and $(\aleph, \Omega^*)$ exists an unique morphisme $u : A \to P$ with $p_i' \circ u = u_i$, for every $i \in I$ then it exists an unique isomorphisme $f : P \to \aleph^I$ with $p_i \circ f = p_i'$, for every $i \in I$.

c) This third step contains the proof of the stated proposition (Problem 3).

As $(\aleph^*, \Omega)$ with $\Omega = (V_d, l_0\}$ is an universal algebra, in accordance with step a) it exists an universal algebra $\left((\aleph^*)^I, \Omega\right)$ with $\Omega = \{\omega_1, \omega_0\}$ defined by:

$$\begin{aligned} \omega_1 &: ((\aleph^*)^I)^2 \to (\aleph^*)^I \text{ by every } a = \{a_i\}_{i\in I} \text{ and } b = \{b_i\}_{i\in I} \in (\aleph^*)^I, \\ \omega_1(a, b) &= \{a_i V_d b_i\}_{i\in I} \end{aligned}$$

and

$$\omega_0 : ((\aleph^*)^I)^0 \to (\aleph^*)^I \text{ by } \omega_0(\{\emptyset\}) = \{e_i = 1\}_{i \in I} = e_{\omega_1},$$

the canonical projections being certainly morphismes between $\left((\aleph^*)^I, \Omega\right)$ and $(\aleph^*, \Omega)$.

As $\left(\aleph, \Omega'\right)$ with $\Omega' = \{V, \Psi_0\}$ is an universal algebra, in accordance with step b) it exists an universal algebra $\left(\aleph^I, \Omega'\right)$ with $\Omega' = \{\theta_1, \theta_0\}$ defined by:

$$\theta_1 : (\aleph^I)^2 \to \aleph^I \text{ by every } a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I} \in \aleph^I, \theta_1(a, b) = \{a_i V_d b_i\}_{i \in I}$$

and

$$\theta_0 : (\aleph^I)^0 \to \aleph^I \text{ by } \theta_0(\{\emptyset\}) = \{e_i = 0\}_{i \in I} = e_{\theta_1}$$

The universal algebras $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega'\right)$ are of the same kind

$$\begin{matrix} \omega_1 & \omega_2 \\ 2 & 0 \end{matrix} = \begin{matrix} \theta_1 & \theta_0 \\ 2 & 0 \end{matrix}$$

We use the property of universality for universal algebra $\left(\aleph', \Omega'\right)$: an universal algebra (A, Ω) can be $\left((\aleph^*)^I, \Omega\right)$ because they are the same kind; the morphismes $u_i : A \to \aleph$ from the assumption will be $\bar{s_i} : (\aleph^*)^I \to \aleph^*$ by every $a = \{a_i\}_{i \in I} \in (\aleph^*)^I, \bar{s_j}(a) = \bar{s_j}\left(\{a_i\}_{i \in I}\right) = s(a_j) \iff \bar{s_j} = s \circ \bar{p_j}$ for every $j \in I$ where $s : \aleph^* \to \aleph$ is Smarandache's function and $\bar{p_j} : (\aleph^*)^I \to \aleph^*$ the canonical projections, morphismes between $\left((\aleph^*)^I, \Omega\right)$ and $(\aleph^*, \Omega)$. As s is a morphisme berween $(\aleph^*, \Omega)$ and $\left(\aleph, \Omega'\right)$, $\bar{s_j}$ are morphismes (as a composition of morphismes) for every $j \in I$. The assumptions of the property of universality being provided $\Longrightarrow$ exists an unique $\bar{s} : (\aleph^*)^I \to \aleph^I$ morphism between $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega\right)$ so that $p_j \circ \bar{s} = \bar{s_j} \iff p_j \circ \bar{s} = S \circ \bar{p_j}$, for every $j \in I$. We finish the proof noticing that s is also surjection: $p_j \circ S$ surjection (as a composition of surjections) $\Longrightarrow s$ surjection.

Remark: The proof of the step 3 can be done directly. As the universal algebras from the statement are built, we can define a correspondence $s : (\aleph^*)^I \to (\aleph^*)^I$ by every $a = \{a_i\}_{i \in I} \in (\aleph^*)^I, s(a) = \{S(a_i)\}_{i \in I}$, which is a function, then morphisme between the universal algebra of the same kind $\left((\aleph^*)^I, \Omega\right)$ and $\left(\aleph^I, \Omega'\right)$ and is also surjective, the required conditions being satisfied evidently.

The stated Problem finds a prolongation s of the Smarandache function S to more comlexe sets (for $I = \{1\} \Rightarrow s = S$). The properties of the function s for the limitation to $\aleph^*$ could bring new properties for the Smarandache function.

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A LINEAR COMBINATION WITH SMARANDACHE FUNCTION TO OBTAIN THE IDENTITY[1]

by

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In this paper we consider a numerical function $i_p: N^* \to N$ (p is an arbitrary prime number) associated with a particular Smarandache Function $S_p: N^* \to N$ such that $(1/p)S_p(a) + i_p(a) = a$.

1. INTRODUCTION. In [7] is defined a numerical function $S: N^* \to N$, $S(n)$ is the smallest integer such that $S(n)!$ is divisible by n. This function may be extended to all integers by defining $S(-n) = S(n)$.

If a and b are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$, and if $[a, b]$ is the last common multiple of a and b then $S([a \cdot b]) = \max\{S(a), S(b)\}$.

Suppose that $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ is the factorization of n into primes. In this case,

$$S(n) = \max\{S(p_i^{a_i}) | i = 1, \dots, r\} \tag{1}$$

Let $a_n(p) = (p^n - 1)/(p-1)$ and $[p]$ be the generalized numerical scale generated by $(a_n(p))_{n \in N}$:

$$[p]: a_1(p), a_2(p), \dots, a_n(p), \dots$$

By (p) we shall note the standard scale induced by the net $b_n(p) = p^n$:

$$(p): 1, p, p^2, p^3, \dots, p^n, \dots$$

In [2] it is proved that

$$S(p^a) = p\left(a_{[p]}\right)_{[p]} \tag{2}$$

That is the value of $S(p^a)$ is obtained multiplying by p the number obtained writing the exponent a in the generalized scale [p] and "reading" it in the standard scale (p).

Let us observe that the calculus in the generalized scale [p] is different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1 \quad \text{and} \quad b_{n+1}(p) = pb_n(p) \tag{3}$$

We have also

$$a_m(p) \le a \Leftrightarrow (p^m - 1)/(p-1) \le a \Leftrightarrow p^m \le (p-1) \cdot a + 1 \Leftrightarrow m \le \log_p((p-1) \cdot a + 1)$$

so if

$$a_{[p]} = v_t a_t(p) + v_{t-1} a_{t-1}(p) + \dots + v_1 a_1(p) = \overline{v_t v_{t-1} \dots v_{1[p]}}$$

is the expression of a in the scale [p] then t is the integer part of $\log_p((p-1) \cdot a + 1)$

$$t = \left[\log_p((p-1) \cdot a + 1)\right]$$

and the digit v_t is obtained from $a = v_t a_t(p) + r_{t-1}$.

In [1] it is proved that

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$$S(p^a)=(p-1)\cdot a+\sigma_{[p]}(a) \tag{4}$$

where $\sigma_{[p]}(a)=v_1+v_2+...+v_v$.

A Legendre formula asert that

$$a!=\prod_{\substack{p_i\le a\\ p_i\ prim}} p_i^{E_{p_i}(a)}$$

where $E_p(a)=\sum_{j\ge 1}\left[\frac{a}{p^j}\right]$.

We have also that ([5])

$$E_p(a)=\frac{\left(a-\sigma_{[p]}(a)\right)}{p-1} \tag{5}$$

and ([1]) $E_p(a)=\left(\left[\frac{a}{p}\right]_{(p)}\right)_{[p]}$.

In [1] is given also the following relation between the function E_p and the Smarandache function

$$S(p^a)=\frac{(p-1)^2}{p}\left(E_p(a)+a\right)+\frac{p-1}{p}\sigma_{[p]}(a)+\sigma_{[p]}(a)$$

There exist a great number of problems concerning the Smarandache function. We present some of these problem.

P. Gronas find ([3]) the solution of the diophantine equation $F_S(n)=n$, where $F_S(n)=\sum_{d|n}S(d)$. The solution are n=9, n=16 or n=24, or n=2p, where p is a prime number.

T. Yau ([8]) find the triplets which verifies the Fibonacci relationship

$$S(n)=S(n+1)+S(n+2).$$

Checking the first 1200 numbers, he find just two triplets which verifies this relationship: (9,10,11) and (119,120,121). He can't find theoretical proof.

The following conjecture that: "the equation $S(x)=S(x+1)$, has no solution", was not completely solved until now.

2. The Function $i_p(a)$. In this section we shall note $S(p^a)=S_p(a)$. From the Legendre formula it results ([4]) that

$$S_p(a)=p\left(a-i_p(a)\right) \text{ with } 0\le i_p(a)\le\left[\frac{a-1}{p}\right]. \tag{6}$$

That is we have

$$\frac{1}{p}S_p(a)+i_p(a)=a \tag{7}$$

and so for each function S_p there exists a function i_p such that we have the linear combination (7) to obtain the identity.

In the following we keep out some formulae for the calculus of i_p. We shall obtain a duality relation between i_p and E_p.

Let $a_{(p)}=\overline{u_k u_{k-1}.....u_1u_0}=u_kp^k+u_{k-1}p^{k-1}+.....+u_1p+u_0$.

Then

$$a=(p-1)\left(u_k\frac{p^k-1}{p-1}+u_{k-1}\frac{p^{k-1}-1}{p-1}+....+u_1\frac{p-1}{p-1}\right)+(u_k+u_{k-1}+...+u_1)+u_0=$$

$$(p-1)\left(\left[\frac{a}{p}\right]_{(p)}\right)_{[p]}+\sigma_{(p)}(a)=(p-1)E_{p(a)}+\sigma_{(p)}(a) \tag{8}$$

From (4) it results

$$a=\frac{S_p(a)-\sigma_{[p]}(a)}{p-1} \tag{9}$$

From (8) and (9) we deduce

$$(p-1)E_p(a)+\sigma_{(p)}(a)=\frac{S_p(a)-\sigma_{[p]}(a)}{p-1}.$$

So,

$$S_p(a)=(p-1)^2E_p(a)+(p-1)\sigma_{(p)}(a)+\sigma_{[p]}(a) \tag{10}$$

From (4) and (7) it results

$$i_p(a)=\frac{a-\sigma_{[p]}(a)}{p} \tag{11}$$

and it is easy to observe a complementary with the equality (5).
Combining (5) and (11) it results

$$i_p(a)=\frac{(p-1)E_p(a)+\sigma_{(p)}(a)-\sigma_{[p]}}{p} \tag{12}$$

From

$$a=\overline{\upsilon_t\upsilon_{t-1}...\upsilon_{1[p]}}=\upsilon_t(p^{t-1}+p^{t-2}+.....+p+1)+\upsilon_{t-1}(p^{t-2}+p^{t-3}+.....+p+1)+$$
$$+.....+\upsilon_2(p+1)+\upsilon_1$$

it results that

$$a=(\upsilon_t p^{t-1}+\upsilon_{t-1}p^{t-2}+.....+\upsilon_2 p+\upsilon_1)+\upsilon_t(p^{t-2}+p^{t-1}+....+1)+\upsilon_{t-1}(p^{t-3}+p^{t-4}+...+1)+.....+$$
$$\upsilon_3(p+1)+\upsilon_2=\left(a_{[p]}\right)_{(p)}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

because

$$\left[\frac{a}{p}\right]=\left[\upsilon_t(p^{t-2}+p^{t-3}+...+p+1)+\frac{\upsilon_t}{p}+\upsilon_{t-1}(p^{t-3}+p^{t-4}+....+p+1)+\frac{\upsilon_{t-1}}{p}+....+\right.$$
$$\left.+\upsilon_3(p+1)+\frac{\upsilon_3}{p}+\upsilon_2+\frac{\upsilon_2}{p}+\frac{\upsilon_1}{p}\right]=\upsilon_t(p^{t-2}+p^{t-3}+...+p+1)+$$
$$+\upsilon_{t-1}(p^{t-3}+p^{t-4}+...+p+1)+...+\upsilon_3(p+1)+\upsilon_2+\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

we have $[n+x]=n+[x]$.
Then

$$a=\left(a_{[p]}\right)_{(p)}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{13}$$

or

$$a=\frac{S_p(a)}{p}+\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]$$

It results that

$$S_p(a)=p\left(a-\left(\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right]\right)\right) \tag{14}$$

From (11) and (14) we obtain

$$i_p(a)=\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{15}$$

It is know that there exists $m,n\in N$ such that the relation

$$\left[\frac{m-n}{p}\right]=\left[\frac{m}{p}\right]-\left[\frac{n}{p}\right] \tag{16}$$

is not verifies.

But if $\frac{m-n}{p}\in N$ then the relation (16) is satisfied.

From (11) and (15) it results

$$\left[\frac{a-\sigma_{[p]}(a)}{p}\right]=\left[\frac{a}{p}\right]-\left[\frac{\sigma_{[p]}(a)}{p}\right].$$

This equality results also by the fact that $i_p(a)\in N$.

From (2) and (11) or from (13) and (15) it results that

$$i_p(a)=a-\left(a_{[p]}\right)_{(p)} \tag{17}$$

From the condition on i_p in (6) it results that $\Delta=\left[\frac{a-1}{p}\right]-i_p(a)\geq 0$.

To calculate the difference $\Delta=\left[\frac{a-1}{p}\right]-i_p(a)$ we observe that

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{a-1}{p}\right]-\left[\frac{a}{p}\right]+\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{18}$$

For $a\in[kp+1,kp+p-1]$ we have $\left[\frac{a-1}{p}\right]=\left[\frac{a}{p}\right]$ so

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{\sigma_{[p]}(a)}{p}\right] \tag{19}$$

If $a=kp$ then $\left[\frac{a-1}{p}\right]=\left[\frac{kp-1}{p}\right]=\left[k-\frac{1}{p}\right]=k-1$ and $\left[\frac{a}{p}\right]=k$.

So, (18) becomes

$$\Delta=\left[\frac{a-1}{p}\right]-i_p(a)=\left[\frac{\sigma_{[p]}(a)}{p}\right]-1 \tag{20}$$

Analogously, if $a=kp+p$, we have

$$\left[\frac{a-1}{p}\right]=\left[\frac{p(k+1)-1}{p}\right]=\left[k+1-\frac{1}{p}\right]=k \quad \text{and} \quad \left[\frac{a}{p}\right]=k+1$$

so, (18) has the form (20).

For any number a, for which Δ is given by (19) or by (20), we deduce that Δ is maximum when $\sigma_{[p]}(a)$ is maximum, so when

$$a_M=\overline{\underbrace{(p-1)(p-1)\ldots(p-1)}_{t \text{ terms}}p}_{[p]} \tag{21}$$

That is

$$a_M=(p-1)a_t(p)+(p-1)a_{t-1}(p)+....+(p-1)a_2(p)+p=$$

$$=(p-1)\left(\frac{p^t-1}{p-1}+\frac{p^{t-1}-1}{p-1}+...+\frac{p^2-1}{p-1}\right)+p=$$

$$=(p^t+p^{t-1}+...+p^2+p)-(t-1)=pa_t(p)-(t-1)$$

It results that a_M is not multiple of p if and only if $t-1$ is not a multiple of p.
In this case $\sigma_{[p]}(a)=(t-1)(p-1)+p=pt-t+1$ and

$$\Delta=\left[\frac{\sigma_{[p]}(a)}{p}\right]=\left[t-\frac{t-1}{p}\right]=t-\left[\frac{t-1}{p}\right].$$

So $i_p(a_M)\geq\left[\frac{a_M-1}{p}\right]-t$ or $i_p(a_M)\in\left[\left[\frac{a_M-1}{p}\right]-t,\left[\frac{a_M-1}{p}\right]\right]$. If $t-1\in(kp,kp+p)$ then $\left[\frac{t-1}{p}\right]=k$ and $k(p-1)+1<\Delta(a_M)<k(p-1)+p+1$ so $\lim_{a_M\to\infty}\Delta(a_M)=\infty$.

We also observe that

$$\left[\frac{a_M-1}{p}\right]=a_t(p)-\left[\frac{t-1}{p}\right]=\frac{p^{t+1}-1}{p-1}-\left[\frac{t-1}{p}\right]\in\left[\frac{p^{kp+1}-1}{p-1}-k,\frac{p^{kp+p+1}-1}{p-1}-k\right].$$

Then if $a_M\to\infty$ (as p^x), it results that $\Delta(a_M)\to\infty$ (as x).

From $\frac{i_p(a_M)}{\left[\frac{a_M-1}{p}\right]}=\frac{a_t(p)-t}{a_t(p)-\left[\frac{t-2}{p}\right]}\to 1$ it results $\lim_{a\to\infty}\frac{i_p(a)}{[a-1]p}=1$.

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EXAMPLES OF SMARANDACHE MAGIC SQUARES

by

M.R. Mudge

For $n \geq 2$, let A be a set of n^2 elements, and l a n-ary law defined on A.

As a generalization of the XVI-th - XVII-th centuries magic squares, we present the *Smarandache magic square of order n*, which is: 2 square array of rows of elements of A arranged so that the law l applied to each horizontal and vertical row and diagonal give the same result.

If A is an arithmetical progression and l the addition of n numbers, then many magic squares have been found. Look at Durer's 1514 engraving "Melancholia" 's one:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

1. Can you find a such magic square of order at least 3 or 4, when A is a set of prime numbers and l the addition?
2. Same question when A is a set of square numbers, or cube numbers, or special numbers [for example: Fibonacci or Lucas numbers, triangular numbers, Smarandache quotients (i.e. q(m) is the smallest k such that mk is a factorial), etc.].

A similar definition for the *Smarandache magic cube of order n*, where the elements of A are arranged in the form of a cube of lenth n:

a. either each element inside of a unitary cube (that the initial cube is divided in)
b. either each element on a surface of a unitary cube
c. either each element on a vertex of a unitary cube.

3. Study similar questions for this case, which is much more complex.

An interesting law may be $l(a_1, a_2, \ldots, a_n) = a_1 + a_2 - a_3 + a_4 - a_5 + \ldots$

Now some examples of Smarandache Magic Squares: if A is a set of PRIME NUMBERS and l is the operation of addition, for orders at least 3 or 4.

Some examples, with the constant in brackets, elements drawn from the first hundred PRIME NUMBERS are :

83	89	41
29	71	113
101	53	59

(213)

101	491	251
431	281	131
311	71	461

(843)

71	461	311
521	281	41
251	101	491

(843)

113	149	257
317	173	29
89	197	233

(519)

97	907	557	397	197
367	167	67	877	677
997	647	337	137	37
107	157	967	617	307
587	277	227	127	937

(2155)

Now recall the year A.D. 1987 and consider the following .. all elements are primes congruent to seven modulo ten ....

967	<u>1987</u>	2017
2707	1657	607
1297	1327	2347

(4971)

<u>1987</u>	9907	11677	5237
4877	12037	9547	2347
10627	2707	4517	10957
11317	4157	3067	10267

(28808)

7	2707	5237	937	947
4157	1297	227	1087	3067
1307	1447	<u>1987</u>	4517	577
2347	3797	1657	1667	367
2017	587	727	1627	4877

(9835)

What about the years 1993, 1997, & 1999 ?

In Personal Computer World, May 1991, page 288, I examine:
A multiplication magic square such as:

18	1	12
4	6	9
3	36	2

with constant 216 obtained by multiplication of the elements in any row/column/principal diagonal.

A geometric magic square is obtained using elements which are a given base raised to the powers of the corresponding elements of a magic square .. it is clearly a multiplication magic square.
e.g. from

8	1	6
3	5	7
4	9	2

C=15

and base 2 obtain

256	2	64
8	32	128
16	512	4

where $M = 2^{15} = 32768$

Note that Henry Nelson of California has found an order three magic square consisting of *consecutive ten-digit prime numbers*. But "How did he do that" ???

A particular case:
TALISMAN MAGIC SQUARES are a relatively new concept, contain the integers from 1 to n^2 in such a way that the difference between any integer and its neighbours (either row-, column- or diagonal-wise) is greater than some given constant, D say.
e.g.

5	15	9	12
10	1	6	3
13	16	11	14
2	8	4	7

illustrates D=2.

References

1. Smarandache, Florentin, "Properties of Numbers", University of Craiova Archives, 1975;
   (see also Arizona State University, Special Collections, Tempe, AZ, USA)
2. Mudge, Mike, England , Letter to R. Muller, Arizona, August 8, 1995.

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Base Solution
(The Smarandache Function)

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Definition of the Smarandache function S(n)

S(n) = the smallest positive integer such that S(n)! is divisible by n.

Problem A: Ashbacher's problem

For what triplets n, n-1, n-2 does the Smarandache function satisfy the Fibonacci reccurrence: S(n)=S(n-1)+S(n-2). Solutions have been found for n=11, 121, 4902, 26245, 32112, 64010, 368140 and 415664. Is there a pattern that would lead to the proof that there is an infinite family of solutions?

The next three triplets n, n-1, n-2 for which the Smarandache funtion S(n) satisfies the relation S(n)=S(n-1)+S(n-2) occur for n=2091206, n=2519648 and n=4573053 . Apart from the triplet obtained from n=26245 the triplets have in common that one member is 2 times a prime and the other two members are primes.

This leads to a search for triplets restricted to integers which meet the following requirements:

$$n = xp^a \text{ with } a \le p+1 \text{ and } S(x) < ap \tag{1}$$

$$n-1 = yq^b \text{ with } b \le q+1 \text{ and } S(y) < bq \tag{2}$$

$$n-2 = zr^c \text{ with } c \le r+1 \text{ and } S(z) < cr \tag{3}$$

p,q and r are primes. With then have S(n)=ap, S(n-1)=bq and S(n-2)=cr. From this and by subtracting (2) from (1) and (3) from (2) we get

$$ap = bq + cr \tag{4}$$

$$xp^a - yq^b = 1 \tag{5}$$

$$yq^b - zr^c = 1 \tag{6}$$

Each solution to (4) generates infinitely many solutions to (5) which can be written in the form:

$$x = x_0 + q^b t, \qquad y = y_0 - p^a t \tag{5'}$$

where t is an integer and (x_0,y_0) is the principal solution, which can be otained using Euclid's algorithm.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for z.

$$z = (yq^b - 1)/r^c \tag{6'}$$

Implementation:
Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and (a,b,c)=(1,1,2) with the parameter t restricted to the interval $-9 \leq t \leq 10$. The output is presented on page 5. Since the correctness of these calculations are easily verfied from factorisations of S(n), S(n-1), and S(n-2) some of these are given in an annex. This study strongly indicates that the set of solutions is infinite.

Problem B: Radu's problem

Show that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1).

The immediate question is what would be this finite set? I order to examine this the following more stringent problem (which replaces "between" with the requirement that S(n) and S(n+1) must also be composite) will be considered.

Find the set of consecutive integers n and n+1 for which two consecutive primes p_k and p_{k+1} exists so that $p_k < Min(S(n),S(n+1))$ and $p_{k+1} > Max(S(n),S(n+1))$.

Consider

$$n+1 = xp_r^s$$

$$n = yp_{r+1}^{s}$$

where p_r and p_{r+1} are consecutive primes. Subtract

$$xp_r^s - yp_{r+1}^s = 1 \qquad (1)$$

The greatest common divisor $(p_r^s, p_{r+1}^s) = 1$ divides the right hand side of (1) which is the condition for this diophantine equation to have infinitely many integer solutions. We are interested in positive integer solutions (x,y) such that the following conditions are met.

I. $S(n+1) = sp_r$, i.e $S(x) < sp_r$

II. $S(n) = sp_{r+1}$, i.e $S(y) < sp_{r+1}$

in addition we require that the interval

III. $sp_r^s < q < sp_{r+1}^s$ is prime free, i.e. q is not a prime.

Euclid's algorithm has been used to obtain principal solutions (x_0, y_0) to (1). The general set of solutions to (1) are then given by

$$x = x_0 + p_{r+1}^s t, \qquad y = y_0 - p_r^s t$$

with t an integer.

Implementation:
The above algorithms have been implemented for various values of the parameters $d = p_{r+1} - p_r$, s and t. A very large set of solutions was obtained. There is no indication that the set would be finite. A pair of primes may produce several solutions. Within the limits set by the design of the program the largest prime difference for which a solution was found is d=42 and the largest exponent which produced solutions is 4. Some numerically large examples illustrating the above facts are given on page 6.

Problem C: Stuparu's problem

Consider numbers written in Smarandache Prime Base 1,2,3,5,7,11,.... given the example that 101 in Smarandache base means $1 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 = 4_{10}$.

As this leads to several ways to translate a base 10 number into a Base Smarandache number it seems that further precisions are needed. Example

$$111_{\text{Smarandache}} = 1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 = 6_{10}$$

$$1001_{\text{Smarandache}} = 1 \cdot 5 + 0 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 = 6_{10}$$

Equipment and programs

Computer programs for this study were written in UBASIC ver. 8.77. Extensive use was made of NXTPRM(x) and PRMDIV(n) which are very convenient although they also set an upper limit for the search routines designed in the main program. Programs were run on a dtk 486/33 computer. Further numerical outputs and program codes are available on request.

Smarandache - Ashbacher's problem.

#	N	S(N)	S(N-1)	S(N-2)	t
1	11	11	5	6	0
2	121	22	5	17	0
3	4902	43	29	14	-4
4	32112	223	197	26	-1
5	64010	173	46	127	-1
6	368140	233	82	151	-1
7	415664	313	167	146	-8
8	2091206	269	202	67	-1
9	2519648	1109	202	907	0
10	4573053	569	106	463	-3
11	7783364	2591	202	2389	0
12	79269727	2861	2719	142	10
13	136193976	3433	554	2879	-1
14	321022289	7589	178	7411	5
15	445810543	1714	761	953	-1
16	559199345	1129	662	467	-5
17	670994143	6491	838	5653	-1
18	836250239	9859	482	9377	1
19	893950202	2213	2062	151	0
20	1041478032	2647	1286	1361	-1
21	1148788154	2467	746	1721	3
22	1305978672	5653	1514	4139	0
23	1834527185	3671	634	3037	-5
24	2390706171	6661	2642	4019	0
25	2502250627	2861	2578	283	-1
26	3969415464	5801	1198	4603	-2
27	3970638169	2066	643	1423	-6
28	6493607750	3049	1262	1787	5
29	6964546435	2161	1814	347	-4
30	11329931930	3023	2026	997	-4
31	13429326313	4778	1597	3181	1
32	13849559620	6883	2474	4409	1
33	14988125477	3209	2986	223	2
34	17560225226	4241	3118	1123	-2
35	25184038673	5582	1951	3631	-2
36	69481145903	6301	3722	2579	3
37	155205225351	8317	4034	4283	-5
38	196209376292	7246	3257	3989	-5
39	344645609138	7226	2803	4423	9
40	401379101876	32122	653	31469	2
41	484400122414	16811	12658	4153	-1
42	533671822944	21089	18118	2971	0
43	561967733244	21722	7159	14563	-1
44	703403257356	13147	10874	2273	-2
45	859525157632	14158	3557	10601	-5
46	898606860813	19973	13402	6571	1
47	1185892343342	18251	12022	6229	-2
48	1188795217601	29242	13049	16193	0
49	1294530625810	17614	5807	11807	-3
50	1517767218627	11617	8318	3299	-8
51	2677290337914	33494	3631	29863	-3
52	3043063820555	14951	12202	2749	5
53	6344309623744	22978	7451	15527	6
54	16738688950356	30538	6977	23561	10
55	19448047080036	34186	17027	17159	-4

Ashbacher's problem (ASHEDIT.UB), 951206, Henry Ibstedt

q=p(j), p=p(j+1), d=p-q, N=x*q's or y*p's, k solutions to x*q's - y*p's = +/-1 will examined.

Parameters for this run: d = 2 , s = 2 , k = 15 .

x	y	q	p	x*q's	y*p's
13039	12198	59	61	45388759	45388758
1876	1755	59	61	6530356	6530355
7975	7544	71	73	40201975	40201976
26	25	101	103	265226	265225
5913	5698	107	109	67697937	67697938
113967	110968	149	151	2530181367	2530181368
38	37	149	151	843638	843637
49063	48438	311	313	4745422423	4745422422
636720	628609	311	313	61584195120	61584195121
60988	60291	347	349	7343504092	7343504091
182614	180527	347	349	21988369126	21988369127
1071729	1062490	461	463	227764918809	227764918810
116	115	461	463	24652436	24652435
214485	212636	461	463	45582566685	45582566684
1071961	1062720	461	463	227814223681	227814223680
131	130	521	523	35558771	35558770
1914834	1900217	521	523	519764455794	519764455793
143	142	569	571	46297823	46297822
3386439	3370000	821	823	2282598729999	2282598730000
206	205	821	823	138852446	138852445
2709522	2696369	821	823	1826328918402	1826328918401
215	214	857	859	157906535	157906534
1475977	1469112	857	859	1084029831673	1084029831672
3689620	3672459	857	859	2709837719380	2709837719379
221	220	881	883	171531581	171531580
2339288	2328703	881	883	1815664113368	1815664113367
5649579	5628340	1061	1063	6359849721459	6359849721460
266	265	1061	1063	299441786	299441785
5650111	5628870	1061	1063	6360448605031	6360448605030
597051	594868	1091	1093	710658461331	710658461332
664416	662113	1151	1153	880218981216	880218981217
1993825	1986914	1151	1153	2641421353825	2641421353826
7311461	7286118	1151	1153	9686230844261	9686230844262
9970279	9935720	1151	1153	13208635589479	13208635589480
8488719	8462680	1301	1303	14368014268119	14368014268120
5093101	5077478	1301	1303	8620587845701	8620587845702
326	325	1301	1303	551787926	551787925
5093753	5078128	1301	1303	8621691421553	8621691421552
8489371	8463330	1301	1303	14369117843971	14369117843970
2617231	2609312	1319	1321	4553356421791	4553356421792
1393198	1389861	1667	1669	3871542597022	3871542597021
9749046	9725695	1667	1669	27091516689894	27091516689895
14432580	14398621	1697	1699	41563073777220	41563073777221
2886176	2879385	1697	1699	8311635620384	8311635620385
425	424	1697	1699	1223918825	1223918824
431	430	1721	1723	1276553471	1276553470
2969160	2962271	1721	1723	8794179823560	8794179823559
1600708	1597131	1787	1789	5111651305252	5111651305251
1599813	1596238	1787	1789	5108793239997	5108793239998
24003460	23949821	1787	1789	76651905056740	76651905056741
19295178	19253993	1871	1873	67545491209098	67545491209097
1753596	1749853	1871	1873	6138710055036	6138710055037
470	469	1877	1879	1655870630	1655870629
14123034	14092985	1877	1879	49757270653386	49757270653385
7612314	7596715	1949	1951	28916143572714	28916143572715
3805913	3798114	1949	1951	14457144927713	14457144927714
488	487	1949	1951	1853717288	1853717287
19032493	18993492	1949	1951	72296846942293	72296846942292
11987503	11963528	1997	1999	47806269851527	47806269851528
500	499	1997	1999	1994004500	1994004499
521	520	2081	2083	2256222281	2256222280

Smarandache - Radu's problem.
q=p(j), p=p(j+1), d=p-q, N=x*q's or y*p's, k solutions to x*q's - y*p's = +/-1 will examined.

Parameters for this run: d = 2 , s = 2 , k = 15 .

x	y	q	p	x*q's	y*p's
4339410	4331081	2081	2083	18792079709010	18792079709009
11162451	11141330	2111	2113	49743464802771	49743464802770
2232913	2228688	2111	2113	9950577093073	9950577093072
2231856	2227633	2111	2113	9945866761776	9945866761777
15626163	15596596	2111	2113	69635198326323	69635198326324
33485239	33421880	2111	2113	149220973745719	149220973745720
35091287	35028624	2237	2239	175602730575503	175602730575504
10025682	10007779	2237	2239	50170207068258	50170207068259
560	559	2237	2239	2802334640	2802334639
2574748	2570211	2267	2269	13232374074172	13232374074171
38612140	38544101	2267	2269	198438946368460	198438946368461
22714160	22676049	2381	2383	128770230019760	128770230019761
596	595	2381	2383	3378819956	3378819955
17036663	17008078	2381	2383	96583585449743	96583585449742
16809771	16783850	2591	2593	112848716268651	112848716268650
21210178	21178283	2657	2659	149736411907522	149736411907523
665	664	2657	2659	4694666585	4694666584
14141227	14119962	2657	2659	99832099049323	99832099049322
10846754	10830625	2687	2689	78313227630626	78313227630625
40482708	40423043	2711	2713	297528512582868	297528512582867
11041232	11024959	2711	2713	81147766449872	81147766449871
683	682	2729	2731	5086602203	5086602202
52209210	52132769	2729	2731	388825011131610	388825011131609
39283344	39227305	2801	2803	308201442969744	308201442969745
701	700	2801	2803	5499766301	5499766300
23571128	23537503	2801	2803	184929665407928	184929665407927
31427937	31383104	2801	2803	246571053955137	246571053955136
4871101	4864860	3119	3121	47386854775261	47386854775260
68783872	68699319	3251	3253	726976811951872	726976811951871
5291818	5285313	3251	3253	55929229733818	55929229733817
5290191	5283688	3251	3253	55912033969191	55912033969192
79364254	79266695	3251	3253	838800879890254	838800879890255
815	814	3257	3259	8645559935	8645559934
53106220	53041059	3257	3259	563353383964780	563353383964779
5687721	5680978	3371	3373	64633219552161	64633219552162
866	865	3461	3463	10373399186	10373399185
890	889	3557	3559	11260501610	11260501609
64188549	64116910	3581	3583	823125773602989	823125773602990
38512771	38469788	3581	3583	493870868197531	493870868197532
896	895	3581	3583	11489910656	11489910655
6744546	6737203	3671	3673	90891127331586	90891127331587
46074703	46027688	3917	3919	706919053836967	706919053836968
980	979	3917	3919	15036031220	15036031219
983	982	3929	3931	15174611303	15174611302
15453744	15438023	3929	3931	238560079731504	238560079731503
30906505	30875064	3929	3931	477104984851705	477104984851704
48071026	48023003	4001	4003	769521032279026	769521032279027
1001	1000	4001	4003	16024009001	16024009000
48073028	48025003	4001	4003	769553080297028	769553080297027
25129997	25105444	4091	4093	420582691321157	420582691321156
25127950	25103399	4091	4093	420548432153950	420548432153951
125643844	125521085	4091	4093	2102810679104164	2102810679104165
8525353	8517096	4127	4129	145204912066537	145204912066536
8523288	8515033	4127	4129	145169740720152	145169740720153
76717852	76643549	4127	4129	1306668351866908	1306668351866909
1055	1054	4217	4219	18761158895	18761158894
89000860	88916499	4217	4219	1582710214456540	1582710214456539
54008086	53957183	4241	4243	971393809450966	971393809450967
1061	1060	4241	4243	19083231941	19083231940
97813539	97725100	4421	4423	1911789192817899	1911789192817900
19561823	19544136	4421	4423	382340544934343	382340544934344
1106	1105	4421	4423	21617036546	21617036545

Smarandache - Radu's problem.
q=p(j), p=p(j+1), d=p-q, N=x*q's or y*p's, k solutions to x*q's - y*p's = +/-1 will examined.

Parameters for this run: d = 2 , s = 2 , k = 15 .

x	y	q	p	x*q's	y*p's
19564035	19546346	4421	4423	382383779007435	382383779007434
97815751	97727310	4421	4423	1911832426890991	1911832426890990
1130	1129	4517	4519	23055716570	23055716569
113814843	113714786	4547	4549	2353145666327187	2353145666327186
10347838	10338741	4547	4549	213943713348142	213943713348141
10345563	10336468	4547	4549	213896677247667	213896677247668
113812568	113712513	4547	4549	2353098630226712	2353098630226713
43262439	43225240	4649	4651	935039789857239	935039789857240
1163	1162	4649	4651	25136152763	25136152762
86528367	86453966	4649	4651	1870154988172767	1870154988172766
1181	1180	4721	4723	26321940221	26321940220
12168478	12158613	4931	4933	295873634303758	295873634303757
36500500	36470909	4931	4933	887500933880500	887500933880501
158172945	158044714	4931	4933	3845937354341145	3845937354341146
86419606	86350053	4967	4969	2132065790970934	2132065790970933
37035199	37005392	4967	4969	913698690661711	913698690661712
125549352	125449153	5009	5011	3150043411177512	3150043411177513
100439231	100359072	5009	5011	2520028441367711	2520028441367712
1253	1252	5009	5011	31437871493	31437871492
75331616	75271495	5009	5011	1890076347300896	1890076347300895
176612447	176471832	5021	5023	4452477674959127	4452477674959128
1256	1255	5021	5023	31664313896	31664313895
117092180	117000379	5099	5101	3044373378656180	3044373378656179
13011376	13001175	5099	5101	338293186736176	338293186736175
13008825	12998626	5099	5101	338226861243825	338226861243826
15979618	15968313	5651	5653	510289941268018	510289941268017
111846018	111766891	5651	5653	3571638481454418	3571668481454419
155004692	154899067	5867	5869	5335523301564788	5335523301564787
51666274	51631067	5867	5869	1778440415416786	1778440415416787
258337240	258161201	5867	5869	8892404132398360	8892404132398361
51880712	51845431	5879	5881	1793134423680392	1793134423680391
17294551	17282790	5879	5881	597745357469191	597745357469190
17291610	17279851	5879	5881	597643708742010	597643708742011
51877771	51842492	5879	5881	1793032774953211	1793032774953212
190222415	190093056	5879	5881	6574589039798015	6574589039798016
224808576	224655697	5879	5881	7769978106009216	7769978106009217
1523	1522	6089	6091	56466627683	56466627682
185502928	185381127	6089	6091	6877691903796688	6877691903796687
1550	1549	6197	6199	59524353950	59524353949
76856752	76807167	6197	6199	2951515167416368	2951515167416367
115284353	115209976	6197	6199	4427242988947577	4427242988947576
1643	1642	6569	6571	70898343323	70898343322
86357725	86305164	6569	6571	3726487909703725	3726487909703724
66555047	66515086	6659	6661	2951202596042207	2951202596042206
1676	1675	6701	6703	75258100076	75258100075
161508670	161413581	6791	6793	7448405321794270	7448405321794269
116589810	116521529	6827	6829	5434009586603490	5434009586603489
69954569	69913600	6827	6829	3260437585177601	3260437585177600
1718	1717	6869	6871	81060670598	81060670597
120719765	120650286	6947	6949	5826033521189885	5826033521189886
127058385	126987104	7127	7129	6453819998221665	6453819998221664
347241454	347051445	7307	7309	18540002175090046	18540002175090045
133551875	133478796	7307	7309	7130634964416875	7130634964416876
80661167	80617174	7331	7333	4335018348995687	4335018348995686
26888278	26873613	7331	7333	1445071808877958	1445071808877957
26884611	26869948	7331	7333	1444874731239771	1444874731239772

Radu's problem, PCW Oct. 95, 951128, Henry Ibstedt

Smarandache - Radu's problem.

q=p(j), p=p(j+1), d=p-q, N=x*q^s or y*p^s. Principal solution to x*q^s - y*p^s = +/-1: x0,y0.
General solutions: x = x0 + t*p^s, y = y0 + t*q^s.

N,N+1	S(N),S(N+1)	d	s	t	q,p
1182293666471533957848301 8	3225562	42	2	-2	1612781
1182293666471533957848301 7	3225646				1612823
1115790649785810026373868363 4	165999	4	3	0	55333
1115790649785810026373868363 5	166011				55337
1754986521322116241350223622 7	165999	4	3	-1	55333
1754986521322116241350223622 6	166011				55337
2703299759212052536347070518228485703913 14	669764	2	4	0	167441
2703299759212052536347070518228485703913 13	669772				167443

Radu's problem (RADUpres.UB), 951129, Henry Ibstedt

Factorisations:

1182293666471533957848301 8 = 2 * 3 * 89 * 193 * 431 * 1612781 ^ 2
1182293666471533957848301 7 = 509 * 3253 * 1612823 ^ 2

1115790649785810026373868363 4 = 2 * 7 * 37 ^ 2 * 56671 * 55333 ^ 3
1115790649785810026373868363 5 = 3 * 5 * 11 * 19 ^ 2 * 16433 * 55337 ^ 3

1754986521322116241350223622 7 = 3 * 11 ^ 2 * 307 * 12671 * 55333 ^ 3
1754986521322116241350223622 6 = 2 * 23 * 37 * 71 * 419 * 743 * 55337 ^ 3

2703299759212052536347070518228485703913 14 = 2 * 3 ^ 3 * 47 * 1289 * 2017 * 119983 * 167441 ^ 4
2703299759212052536347070518228485703913 13 = 37 * 23117 * 24517 * 38303 * 167443 ^ 4

Radufact, 951129, Henry Ibstedt

Adjacent primes:

Smarandache function values in the above examples: S1 and S2.
P1 and P2 are consecutive primes below and above S1 and S2 respectively. Prime gap = G.

P1	S1	S2	P2	G
3225539	3225562	3225646	3225647	108
165983	165999	166011	166013	30
669763	669764	669772	669787	24

Raduadj, 951130, Henry Ibstedt

Factorisations: Ashbacher - Fibonacci

N = 1185892343342 = 2 * 7 ^ 2 * 47 * 14107 * 18251 ^ 1
N-1 = 1185892343341 = 23 * 1427 * 6011 ^ 2
N-2 = 1185892343340 = 2 ^ 2 * 3 * 5 * 523 * 6067 * 6229 ^ 1
S(N) = 18251 = 18251 ^ 1
S(N-1) = 12022 = 2 * 6011 ^ 1
S(N-2) = 6229 = 6229 ^ 1

N = 1188795217601 = 67 * 83 * 14621 ^ 2
N-1 = 1188795217600 = 2 ^ 6 * 5 ^ 2 * 97 * 587 * 13049 ^ 1
N-2 = 1188795217599 = 3 ^ 2 * 11 * 17 * 181 * 241 * 16193 ^ 1
S(N) = 29242 = 2 * 14621 ^ 1
S(N-1) = 13049 = 13049 ^ 1
S(N-2) = 16193 = 16193 ^ 1

N = 1294530625810 = 2 * 5 * 1669 * 8807 ^ 2
N-1 = 1294530625809 = 3 ^ 2 * 101 * 103 * 2381 * 5807 ^ 1
N-2 = 1294530625808 = 2 ^ 4 * 7 * 19 * 67 * 769 * 11807 ^ 1
S(N) = 17614 = 2 * 8807 ^ 1
S(N-1) = 5807 = 5807 ^ 1
S(N-2) = 11807 = 11807 ^ 1

N = 1517767218627 = 3 * 11 * 107 * 163 * 227 * 11617 ^ 1
N-1 = 1517767218626 = 2 * 73 * 601 * 4159 ^ 2
N-2 = 1517767218625 = 5 ^ 3 * 7 * 17 * 157 * 197 * 3299 ^ 1
S(N) = 11617 = 11617 ^ 1
S(N-1) = 8318 = 2 * 4159 ^ 1
S(N-2) = 3299 = 3299 ^ 1

N = 2677290337914 = 2 * 3 * 37 * 43 * 16747 ^ 2
N-1 = 2677290337913 = 479 * 739 * 2083 * 3631 ^ 1
N-2 = 2677290337912 = 2 ^ 3 * 17 ^ 3 * 2281 * 29863 ^ 1
S(N) = 33494 = 2 * 16747 ^ 1
S(N-1) = 3631 = 3631 ^ 1
S(N-2) = 29863 = 29863 ^ 1

N = 3043063820555 = 5 * 11 * 571 * 6481 * 14951 ^ 1
N-1 = 3043063820554 = 2 * 41 * 997 * 6101 ^ 2
N-2 = 3043063820553 = 3 * 53 * 73 * 283 * 337 * 2749 ^ 1
S(N) = 14951 = 14951 ^ 1
S(N-1) = 12202 = 2 * 6101 ^ 1
S(N-2) = 2749 = 2749 ^ 1

N = 6344309623744 = 2 ^ 6 * 751 * 11489 ^ 2
N-1 = 6344309623743 = 3 ^ 3 * 7 ^ 2 * 13 * 31 * 1597 * 7451 ^ 1
N-2 = 6344309623742 = 2 * 107 * 211 * 9049 * 15527 ^ 1
S(N) = 22978 = 2 * 11489 ^ 1
S(N-1) = 7451 = 7451 ^ 1
S(N-2) = 15527 = 15527 ^ 1

N = 16738688950356 = 2 ^ 2 * 3 * 31 * 193 * 15269 ^ 2
N-1 = 16738688950355 = 5 * 197 * 1399 * 1741 * 6977 ^ 1
N-2 = 16738688950354 = 2 * 7 ^ 2 * 19 * 23 * 53 * 313 * 23561 ^ 1
S(N) = 30538 = 2 * 15269 ^ 1
S(N-1) = 6977 = 6977 ^ 1
S(N-2) = 23561 = 23561 ^ 1

N = 19448047080036 = 2 ^ 2 * 3 ^ 2 * 43 ^ 2 * 17093 ^ 2
N-1 = 19448047080035 = 5 * 7 * 19 * 37 * 61 * 761 * 17027 ^ 1
N-2 = 19448047080034 = 2 * 97 * 1609 * 3631 * 17159 ^ 1
S(N) = 34186 = 2 * 17093 ^ 1
S(N-1) = 17027 = 17027 ^ 1
S(N-2) = 17159 = 17159 ^ 1

ON RADU'S PROBLEM

by H. Ibstedt

For a positive integer n, the Smarandache function S(n) is defined as the smallest positive integer such that S(n)! is divisible by n. Radu [1] noticed that for nearly all values of n up to 4800 there is always at least one prime number between S(n) and S(n+1) including possibly S(n) and S(n+1). The exceptions are n=224 for which S(n)=8 and S(n+1)=10 and n=2057 for which S(n)=22 and S(n+1)=21. Radu conjectured that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1). The conjecture does not hold if there are infinitely many solutions to the following problem.

Find consecutive integers n and n+1 for which two consecutive primes p_k and p_{k+1} exist so that p_k < Min(S(n),S(n+1)) and p_{k+1} > Max(S(n),S(n+1)).

Consider

$$n+1 = xp_r^s \tag{1}$$

and

$$n = yp_{r+1}^s \tag{2}$$

where p_r and p_{r+1} are consecutive prime numbers. Subtract (2) form (1).

$$xp_r^s - yp_{r+1}^s = 1 \tag{3}$$

The greatest common divisor $(p_r^s, p_{r+1}^s) = 1$ divides the right hand side of (3) which is the condition for this diophantine equation to have infinitely many integer solutions. We are interested in positive integer solutions (x,y) such that the following conditions are met.

$$S(n+1) = sp_r, \text{ i.e } S(x) < sp_r \tag{4}$$

$$S(n) = sp_{r+1}, \text{ i.e } S(y) < sp_{r+1} \tag{5}$$

In addition we require that the interval

$$sp_r^s < q < sp_{r+1}^s \text{ is prime free, i.e. q is not a prime.} \tag{6}$$

Euclid's algorithm is used to obtain principal solutions (x_0,y_0) to (3). The general set of solutions to (3) are then given by

$$x = x_0 + p_{r+1}^s t, \qquad y = y_0 - p_r^s t \tag{7}$$

with t an integer.

These algorithms were implemented for different values of the parameters $d=p_{r+1}-p_r$, s and t resulted in a very large number of solutions. Table 1 shows the 20 smallest (in respect of n) solutions found. There is no indication that the set would be finite. A pair of primes may produce several solutions.

Table 1. The 20 smallest solutions which ocurred for s=2 and d=2.

#	n	S(n)	S(n+1)	P1	P2	t
1	265225	206	202	199	211	0
2	843637	302	298	293	307	0
3	6530355	122	118	113	127	-1
4	24652435	926	922	919	929	0
5	35558770	1046	1042	1039	1049	0
6	40201975	142	146	139	149	1
7	45388758	122	118	113	127	-4
8	46297822	1142	1138	1129	1151	0
9	67697937	214	218	211	223	0
10	138852445	1646	1642	1637	1657	0
11	157906534	1718	1714	1709	1721	0
12	171531580	1766	1762	1759	1777	0
13	299441785	2126	2122	2113	2129	0
14	551787925	2606	2602	2593	2609	0
15	1223918824	3398	3394	3391	3407	0
16	1276553470	3446	3442	3433	3449	0
17	1655870629	3758	3754	3739	3761	0
18	1853717287	3902	3898	3889	3907	0
19	1994004499	3998	3994	3989	4001	0
20	2256222280	4166	4162	4159	4177	0

Within the limits set by the design of the program the largest prime difference for which a solution was found is d=42 and the largest exponent which produced solutions is s=4. Some numerically large examples illustrating the these facts are given in table 2.

Table 2.

n/n+1	S(n)/ S(n+1)	d	s	t	p_r/p_{r+1}
1182293666471533957848 3018	3225562	42	2	-2	1612781
1182293666471533957848 3017	3225646				1612823
1115790649785810026373868 3634	165999	4	3	0	55333
1115790649785810026373868 3635	166011				55337
1754986521322116241350223 6227	16599	4	3	-1	55333
1754986521322116241350223 6226	166011				55337
2703299759212052536347070518 22848570391314	669764	2	4	0	167441
2703299759212052536347070518 22848570391313	669772				167443

To see the relation between these large numbers and the corresponding values of the Smarandache function in table 2 the factorisations of these large numbers are given below:

$$1182293666471533957848301 8 = 2 \cdot 3 \cdot 89 \cdot 193 \cdot 431 \cdot 1612781^2$$
$$1182293666471533957848301 7 = 509 \cdot 3253 \cdot 1612823^2$$

$$1115790649785810026373868363 4 = 2 \cdot 7 \cdot 37^2 \cdot 56671 \cdot 55333^3$$
$$1115790649785810026373868363 5 = 3 \cdot 5 \cdot 11 \cdot 19^2 \cdot 16433 \cdot 55337^3$$

$$1754986521322116241350223622 7 = 3 \cdot 11^2 \cdot 307 \cdot 12671 \cdot 55333^3$$
$$1754986521322116241350223622 6 = 2 \cdot 23 \cdot 37 \cdot 71 \cdot 419 \cdot 743 \cdot 55337^3$$

$$2703299759212052536347070518228485703913 14 = 2 \cdot 3^3 \cdot 47 \cdot 1289 \cdot 2017 \cdot 119983 \cdot 167441^4$$
$$2703299759212052536347070518228485703913 13 = 37 \cdot 23117 \cdot 24517 \cdot 38303 \cdot 167443^4$$

It is also interesting to see which are the nearast smaller P_k and nearast bigger P_{k+1} primes to $S_1 = \text{Min}(S(n),S(n+1))$ and $S_2 = \text{Max}(S(n),S(n+1))$ respectively. This is shown in table 3 for the above examples.

Table 3.

P_k	S_1	S_2	P_{k+1}	$G=P_{k+1}-P_k$
3225539	3225562	3225646	3225647	108
165983	165999	166011	166013	30
669763	669764	669772	669787	24

Conclusion: There are infintely many intervals {Min(S(n),S(n-1)),Max(S(n),S(n-1))} which are prime free.

References:

I. M. Radu, *Mathematical Spectrum, Sheffield Univeristy, UK*, Vol. 27, No.2, 1994/5, p. 43.

SOME CONVERGENCE PROBLEMS INVOLVING THE SMARANDACHE FUNCTION

by

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In this paper we consider some series attached to the Smarandache function (Dirichlet series and other (numerical) series). Asymptotic behaviour and convergence of these series is established.

1. INTRODUCTION. The Smarandache function $S : \mathcal{N}^* \to \mathcal{N}^*$ is defined [3] such that $S(n)$ is the smallest integer n with the property that n! is divisible by n. If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_t^{\alpha_t} \tag{1.1}$$

is the decomposition into primes of the positiv integer n, then

$$S(n) = \max_i S(p_i^{\alpha_i}) \tag{1.2}$$

and more general if $n_1 \overset{d}{\vee} n_2$ is the smallest commun multiple of n_1 and n_2 then .

$$S(n_1 \overset{d}{\vee} n_2) = \max(S(n_1), S(n_2))$$

Let us observe that on the set N of non-negative integers, there are two latticeal structures generated respectively by $\vee$ = max, $\wedge$ = min and $\overset{d}{\vee}$ = the last commun multiple, $\underset{d}{\wedge}$ = the greatest commun division. if we denote by $\leq$ and $\leq_d$ the induced orders in these lattices, It results

$$S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$

The calculus of $S(p^\alpha)$ depends closely of two numerical scale, namely the standard scale

$$(p) : 1, p, p^2, \ldots, p^n, \ldots$$

and the generalised numerical scale [p]

$$[p] : a_1(p), a_2(p), ..., a_n(p), ...$$

where $a_k(p) = (p^k-1)/(p-1)$. The dependence is in the sens that

$$S(p^{\alpha}) = p(\alpha_{[p]})(p) \tag{1.3}$$

so, $S(p^{\alpha})$is obtained multiplyng p by the number obtained writing α in the scale [p] and "reading" it in the scale (p).

Let us observe that if $b_n(p) = p^n$ then the calculus in the scale [p] is essentialy different from the standard scale (p), because :

$$b_{n+1}(p) = pb_n(p) \quad \text{but } a_{n+1}(p) = pa_n(p) + 1$$

(for more details see [2]).

We have also [1] that

$$S(p^{\alpha}) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{1.4}$$

where $\sigma_{[p]}(\alpha)$is the sum of digits of the number α writen in the scale [p].

In [4] it is showed that if φ is Euler's totient function and we note $S_p(\alpha) = S(p^{\alpha})$ then

$$S_p(p^{\alpha-1}) = \varphi(p^{\alpha}) + p \tag{1.5}$$

It results that $\varphi(p_i^{\alpha_i}) = S(p_i^{p_i^{\alpha_i-1}}) - p$ so

$$\varphi(n) = \prod_{i=1}^{r}\left(S(p_i^{p_i^{\alpha_i-1}}) - p_i\right)..$$

In the same paper [4] the function S is extended to the set Q of rational numbers.

2. GENERATING FUNCTIONS. It is known that we may attashe to each numerical function f:N*-->C the Dirichlet serie :

$$D_f(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \tag{2.1}$$

which for some $z = x + iy$ may be convergent or not.
The simplest Dirichlet series is:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{2.2}$$

called Riemann's function or zeta function where is convergent for $Re(z) > 1$.

It is said for instance that if f is Möbius function ($\mu(1) = 1$, $\mu(p_1 p_2 \cdots p_r) = (-1)^r$ and $\mu(n) = 0$ if n is divisible by the squar of a prime number) then $D_\mu(z) = 1/\zeta(z)$ for $x>1$, and if f is Euler's totient function ($\varphi(n)$ = the number of positive integers not greater than and prime to the positive integer n) then $D_\varphi(z)=\zeta(z-1)/\zeta(z)$ for $x>2$).

We have also $D_d(z) = \zeta^2(z)$, for $x > 1$, where d(n) is the number of divisors of n, including 1 and n, and $D_{\sigma_k}(n) = \zeta(z)\cdot\zeta(z-k)$ (for $x > 1$, $x > k+1$), where $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. We write $\sigma(n)$ for $\sigma_1(n)$.

In the sequel let we suppose that z is a real number, so z = x.

For the Smarandache function we have:

$$D_s(x) = \sum_{n=1}^{\infty} \frac{S(n)}{n^x}$$

If we note :

$$F_f^*(n) = \sum_{k \mid n} f(k)$$

it is said that Möbius function make a connection between f and F_f^* by the inversion formula:

$$f(n) = \sum_{k \mid n} F_f^*(k)\mu\left(\frac{n}{k}\right) \qquad (2.3)$$

The functions F_f^* are also called generating functions.

In [4] the Smarandache functions is regarded as a generating function and is constructed the function s_* such that:

$$s_*(n) = \sum_{k \mid n} S(k)\mu\left(\frac{n}{k}\right)$$

2.1. PROPOSITION. For all x > 2 we have :

(i) $3(x) \le D_S(x) \le 3(x-1)$

(ii) $1 \le D_{\mu}(x) \le D_{\varphi}(x)$

(iii) $3^2(x) \le D_{fS}(x) \le 3(x) \cdot 3(x-1)$

Proof. (i) The asertion results from the fact that $1 \le S(n) \le n$.

(ii) Using the multiplication of Dirichlet series we have:

$$\frac{1}{\zeta(x)} \cdot D_S(x) = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^x}\right)\left(\sum_{n=1}^{\infty} \frac{S(n)}{n^x}\right) = \mu(1)S(1) + \frac{\mu(1)S(2)+\mu(2)S(1)}{2^x} +$$

$$+\frac{\mu(1)S(3)+\mu(2)S(1)}{3^x} + \frac{\mu(1)S(4)+\mu(2)S(2)+\mu(4)S(1)}{4^x} + \ldots = \sum_{n=1}^{\infty} \frac{\varepsilon_0(n)}{n^x} = D_{\varepsilon_0}(x)$$

and the asertion result using (i).

(iii) We have

$$3(x) \cdot D_S(x) = \left(\sum_{n=1}^{\infty} \frac{1}{n^x}\right)\left(\sum_{n=1}^{\infty} \frac{S(n)}{n^x}\right) = S(1) + \frac{S(1)+S(2)}{2^x} + \frac{S(1)+S(3)}{3^x} + \ldots = D_{\varepsilon_1}(x)$$

so the inequalities holds using (i).

Let us observe that (iii) is equivalent to $D_{\varphi}(x) \le D_{\varepsilon_2} < D_{\sigma}(x)$. These inequalities can be deduced also observing that from $1 \le S(n) \le n$ it result:

$$\sum_{k \mid n} 1 \le \sum_{k \mid n} S(k) \le \sum_{k \mid n}$$

so,

$$d(n) \le F_S(n) \le \sigma(n) \qquad (2.4)$$

But from the fact that $F_S < n + 4$ (proved in [5]) we deduce

$$d(n) \le F_S(n) \le n + 4 \qquad (2.5)$$

Until now it is not known a closed formula for the calculus of the functions $D_{\zeta}(x)$, $D_{\varepsilon_3}(x)$ or $D_{\varepsilon_S}(x)$, but we can deduce asimptotic behaviour of these functions using the following well known results:

2.2. THEOREM. (i) $\zeta(z) = \frac{1}{z-1} + O(1)$

(ii) $\ln\zeta(z) = \ln\frac{1}{z-1} + O(z-1)$

(iii) $\zeta'(z) = -\frac{1}{(z-1)^2} + O(1)$

for all complex number.
Then from the proposition 2.1 we can get inequalities as the fallowings:

(i) $\frac{1}{x-1} + O(1) \le D_{\varepsilon}(x) \le \frac{1}{x-2} + O(1)$

(ii) $1 \le D_{\varepsilon_0}(x) \le \frac{x-1}{e^{A(x-2}}$ for some positive constant A

(iii) $-\frac{1}{(x-1)^2} + O(1) \le D'_{\varepsilon}(x) \le -\frac{1}{(x-2)^2} + O(1)$.

The Smarandache functions S may be extended to all the nonnegative integers defining S(-n) = S(n).

In [3] it is proved that the serie

$$\sum_{n=1}^{\infty} \frac{S(n)}{n!}$$

is convergent and has the sum $q \in (e-1,2)$.
We can consider the function

$$f(z) = \sum_{k=1}^{\infty} \frac{S(k)}{(k+1)!} z^k$$

convergent for all $z \in C$ because

$$\frac{a_{n+1}}{a_n} = \frac{S(n+1)}{(n+2)S(n)} \le \frac{n+1}{(n+2)S(n)} \le \frac{1}{S(n)}$$

and so $\frac{a_{n+1}}{a_n} \to 0$

2.3. PROPOSITION. The function f statisfies $|f(z)| \le qz$ an the unit disc $U(0,1) = \{z \mid |z| < 1\}$.

Proof. A lema does to Schwartz asert that if the function f is olomorphe on the unit disc $U(0,1) = \{ z \mid |z| < 1\}$ and satisfies $f(0) = 0$, $|f(z)| \le 1$ for $z \in U(0,1)$ then $|f(z)| \le |z|$ on $U(0,1)$ and $|f'(0)| \le 1$.

For $|z| < 1$ we fave $|f(z)| < q$ so $(1/q)\, f(z)$ satisfies the conditions of Schwartz lema.

3. SERIES INVOLVING THE SMARANDACHE FUNCTION. In this section we shall studie the convergence of some series concerning the function S.

Let b: N*–>N* be the function defined by: b(n) is the complement of n until the smallest factorial. From this definition it results that b(n) = (S(n)!)/n for all $n \in N^*$.

3.1. PROPOSITION. The sequences $(b(n))_{n\geq 1}$ and also $(b(n)/n^k)_{n\geq 1}$ for $k\in R$, are divergent.

Proof. (i) The asertion results from the fact that $b(n!) = 1$ and if $(p_n)_{n\geq 1}$ is the sequence of prime members then

$$b(p_n) = \frac{S(p_n)!}{p_n} = \frac{p_n!}{p_n} = (p_n - 1)!$$

(ii) Let we note $x_n = b(n)/n^k$. Then

$$x_n = \frac{S(n)!}{n^{k+1}}$$

and for $k > 0$ it results

$$x_{n!} = \frac{S(n!)!}{(n!)^{k+1}} = \frac{n!}{(n!)^{k+1}} \to 0$$

$$x_{p_n} = \frac{p_n!}{(p_n)^{k+1}} = \frac{(p-1)!}{(p_n)^{k+1}} > \frac{p_1 \cdot p_2 \cdots p_{n-1}}{p_n^{k+1}} > p_n$$

because it in said [6] that $p_1 p_2 \cdots p_{n-1} > p_n^{k+2}$ for n sufficiently large.

3.2. PROPOSITION. The sequence $T(n) = 1 + \sum_{i=2}^{n} \frac{1}{b(n)} - \ln b(n)$ is divergent.

Proof. If we suppose that $\lim_{n\to\infty} T(n) = l < \infty$, then because $\sum_{i=2}^{\infty} \frac{1}{b(n)} = \infty$ (see [3]) it results the contradiction $\lim_{n\to\infty} \ln b(n) = \infty$.

If we suppose $\lim_{n\to\infty} T(n) = -\infty$, from the equality $\ln b(n) = 1 + \sum_{i=2}^{\infty} \frac{1}{b(n)} - T(n)$ it results $\lim_{n\to\infty} \ln b(n) = \infty$.

We can't have $\lim_{n\to\infty} T(n) = +\infty$ because $T(n) < 0$. Indeed, from $i \leq S(i)!$ for $i \geq 2$ it results

$$i / S(i)! \leq 1 \text{ for all } i \geq 2$$

so

$$T(p_n) = 1 + \frac{2}{S(2)!} + \ldots + \frac{p_n}{S(p_n)!} - \ln((p_n - 1)!) < 1 + (p_n - 1) - \ln((p_n - 1)!) =$$

$$= p_n - \ln((p_n - 1)!).$$

But for k sufficiently large we have $e^k < (k-1)!$ that is there exists $m\in N$ so that $p_n < \ln((p_n - 1)!)$ for $n \geq m$. It results $p_n - \ln((p_n - 1)!) < 0$ for $n \geq m$, and so $T(n) < 0$.

Let now be the function

$$H_b(x) = \sum_{2\leq n\leq x} b(n).$$

3.3. PROPOSITION. The serie

$$\sum_{n\geq 2} H_b^{-1}(n) \qquad (3.1)$$

is convergent.

Proof. the sequence $(b(2)+b(3)+ \ldots + b(n))_n$ is strictley increasing to ∞ and

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}>\frac{S(2)!}{2}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}+\frac{S(7)}{7}>\frac{S(7)!}{7}$$

.....

so we have

$$\sum_{n\geq 2} H_b^{-1}(n)=\frac{1}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}}+\ldots$$

$$\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\ldots+\frac{S(n)!}{n}}+\ldots<$$

$$<\frac{2}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(3)!}{3}}+\frac{2}{\frac{S(5)!}{5}}+\frac{4}{\frac{S(7)!}{7}}+\frac{2}{\frac{S(11)!}{11}}+\ldots+\frac{p_{k+1}-p_k}{\frac{S(p_k)!}{p_k}}+\ldots$$

$$<1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{S(p_k)!}=1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}=1+\frac{1}{2}+\frac{1}{12}+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k}{p_k!}.$$

But $(p_n-1)! > p_1p_2\ldots p_n$ for $n \geq 4$ and then

$$\sum_{n\geq 2} H_b^{-1}(n)<\frac{19}{12}+\sum_{k\geq 4} a_k$$

where $a_k=\frac{p_k(p_{k+1}-p_k)}{p_k!}=\frac{p_{k+1}-p_k}{1\cdot 2\cdot 3\cdot\ldots\cdot(p_k-1)}<\frac{p_{k+1}-p_k}{p_1p_2\ldots p_k}<\frac{p_{k+1}}{p_1p_2\ldots p_k}$

Because $p_1p_2\ldots p_k > p^3_{k+1}$ for k sufficiently large, it results

$$a_k<\frac{p_{k+1}}{p^3_{k+1}}=\frac{1}{p^2_{k+1}} \text{ for } k\geq k_0$$

and the convergence of the serie (3.1) follows from the convergence of the serie $\sum_{k\geq k_0}\frac{1}{p^2_{k+1}}$.

In the followings we give an elementary proof of the convergence of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}$, $\alpha\in R, \alpha>1$ provides information on the convergence behavior of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)!}$.

3.4. PROPOSITION. The series $\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}$, converges if $\alpha\in R$ and $\alpha>1$.

Proof.

$$\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}=\frac{1}{2^{\alpha}\sqrt{2!}}+\frac{1}{3^{\alpha}\sqrt{3!}}+\frac{1}{4^{\alpha}\sqrt{4!}}+\frac{1}{5^{\alpha}\sqrt{5!}}+\frac{1}{3^{\alpha}\sqrt{3!}}+\frac{1}{7^{\alpha}\sqrt{7!}}+\frac{1}{4^{\alpha}\sqrt{4!}}+\ldots\quad\sum_{t=2}^{\infty}\frac{m_t}{t^{\alpha}\sqrt{t!}}$$

where m_t denotes the number of elements of the set

$$M_t\{k\in N^*, S(k)=t\}=\{k\in N^*, k\mid t \text{ and } k\mid (t-1)!\}.$$

It follows that $M_t \{ k \in N^*, k \mid t \}$ and there fore $m_t < d(t!)$.
Hence $m_t < 2\sqrt{t!}$ and consequently we have

$$\sum_{t=2}^{\infty} \frac{m_t}{t^\alpha \sqrt{t!}} < \sum_{t=2}^{\infty} \frac{2\sqrt{t!}}{t^\alpha \sqrt{t!}} = 2 \sum_{t=2}^{\infty} \frac{1}{t^\alpha}$$

So, $\sum_{t=2}^{\infty} \frac{m_t}{t^\alpha \sqrt{t!}}$ converges.

3.5. PROPOSITION. $t^\alpha \sqrt{t!} < t!$ if $\alpha \in R$, $\alpha > 1$ and $t > t_\alpha = [e^{2\alpha+1}]$, $t \in N^*$. (where [x] means the integer part of x).

Proof. $t^\alpha \sqrt{t!} < t! \Leftrightarrow t^{2\alpha} t! < (t!)^2 \Leftrightarrow t^{2\alpha} < t!$ (2)

On the other hand $t^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow (e \cdot \frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha} \cdot (\frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (3)

If $t > e^{2\alpha+1} \Rightarrow (\frac{t}{e})^{t-2\alpha} > (\frac{e^{2\alpha+1}}{e})^{t-2\alpha} = (e^{2\alpha})^{t-2\alpha} > (e^{2\alpha})^{e^{2\alpha+1}-2\alpha}$

Applyng the well-known result that $e^x > 1 + x$ if $x > 0$ for $x = 2\alpha$ we have
$(e^{2\alpha})^{e^{2\alpha+1}-2\alpha} > (e^{2\alpha})^{2\alpha+1+1-2\alpha} = (e^{2\alpha})^2 = e^{4\alpha} > e^{2\alpha}$.
So, if $t > e^{2\alpha+1}$ we fave $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (4)
It is well known that $(\frac{t}{e})^t < t!$ if $t \in N^*$. (5)

Now, the proof of the proposition is obtained as follows:

If $t > t_\alpha = [e^{2\alpha+1}]$, $t \in N^*$ we have $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha} \Leftrightarrow t^{2\alpha} < (\frac{t}{e})^t < t!$. Hence $t^{2\alpha} < t!$ if $t > t_\alpha$ and this proves the proposition.

CONSEQUENCE. The series $\sum_{t=2}^{\infty} \frac{1}{S(t)!}$ converges.

Proof. $\sum_{t=2}^{\infty} \frac{1}{S(t)!} = \sum_{t=2}^{\infty} \frac{m_t}{t!}$ where m_t is defined as above.

If $t > t_\alpha$ we have $t^\alpha \sqrt{t!} < t! \Leftrightarrow \frac{1}{t^\alpha \sqrt{t!}} > \frac{1}{t!} \Leftrightarrow \frac{m_t}{t^\alpha \sqrt{t!}} > \frac{m_t}{t!}$.

Since $\sum_{t=2}^{\infty} \frac{m_t}{t^\alpha \sqrt{t!}}$ converges it results that $\sum_{t=2}^{\infty} \frac{m_t}{t!}$ also converges.

REMARQUE. From the definition of the Smarandache function it results that

card $\{ k \in N^*: S(k)=t \}$ = card $\{ k \in N^*: k \mid t$ and $k \mid (t-1)! \}$ = $d(t!)-d((t-1)!)$

so we get

$$\sum_{t=2}^{n} card(dS^{-1}(t)) = \sum_{k=2}^{n} (d(t!) - d((t-1)!)) = d(n!) - 1$$

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ON THE SMARANDACHE FUNCTION AND THE FIXED - POINT THEORY OF NUMBERS

by

Albert A. Mullin

This brief note points out several basic connections between the Smarandache function, fixed-point theory [1] and prime-number theory. First recall that fixed-point theory in function spaces provides elegent, if not short, proofs of the existence of solutions to many kinds of differential equations, integral equations, optimization problems and game-theoretic problems. Further, fixed-point theory in the ring of rational integers and fixed-lattice-point theory provide many results on the existence of solutions in diophantine theory. Here are four fundamental examples of fixed-point theory in number theory. (1) There is the well-known basic result that for $p>4$, p is prime iff $S(p) = p$. (2) Recall that the present author defined [2] the number-theoretic function $\Psi(n)$ as the *product* of the primes alone in the *mosaic* of n, where the mosaic of n is obtained from n by recursively applying the unique factorization theorem/fundamental theorem of arithmetic to itself! Now the asymptotic density of fixed points of $\Psi(n)$ is $7/\pi^2$, just as the asymptotic density of square-free numbers is $6/\pi^2$. Indeed, (3) the theory of *perfect* numbers is also connected to fixed-point theory, since if one puts $f(n) = \delta(n) - n$, where $\delta(n)$ is the sum of the divisors on n, then n is perfect iff $f(n) = n$. Finally, (4) the present author defined [2] the number-theoretic function $\Psi^*(n)$ as the *sum* of the primes alone in the *mosaic* of n. Here we have a striking similarity to the Smarandache function itself (see example (1) above), since $\Psi^*(n) = n$ iff $n = 4$ or $n = p$ for some prime p; i.e., if > 4, n is prime iff $\Psi^*(n) = n$. Thus, the distribution function for the fixed points of S(n) or of $\Psi^*(n)$ is essentially the distribution function for the primes, $\Pi(n)$.

Problems

(1) Put $S^2(n) = S(S(n))$ and define $S^m(n)$ recursively, where S(n) is the Smarandache function. (*Note*: This approach aligns Smarandache function theory more closely with recursive function theory/computer theory.) For each n, determine the *least* m for which $S^m(n)$ is prime.

(2) Prove that $S(n) = S(n+3)$ for only finitely many n.

(3) Prove that $S(n) = S(n+2)$ for only finitely many n.

(4) Prove that $S(n) = S(n+1)$ for no n.

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ON THE CALCULUS OF SMARANDACHE FUNCTION

by

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Introduction. The Smarandache function $S : \mathbf{N}^* \longrightarrow \mathbf{N}^*$ is defined [5] by the condition that $S(n)$ is the smallest integer m such that $m!$ is divisible by n.So,we have $S(1) = 1, S(2^{12}) = 16$.

Considering on the set $\mathbf{N}^*$ two laticeal structures $\mathcal{N} = (\mathbf{N}^*, \wedge, \vee)$ and $\mathcal{N}_d = (\mathbf{N}^*, \underset{d}{\wedge}, \overset{d}{\vee})$,where $\wedge = \min, \vee = \max$, $\underset{d}{\wedge}$= the greattest common divisor, $\overset{d}{\vee}$= the smallest common multiple,it results that S has the followings properties:

$$(s_1)\ \ S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$
$$(s_2)\ \ n_1 \leq_d n_2 \Longrightarrow S(n_1) \leq S(n_2)$$

where $\leq$ is the order in the lattice $\mathcal{N}$ and $\leq_d$ is the order in the lattice $\mathcal{N}_d$.It is said that

$$n_1 \leq_d n_2 \Longleftrightarrow n_1 \text{ divides } n_2$$

From these properties we deduce that in fact on must consider

$$S : \mathcal{N}_d \longrightarrow \mathcal{N}$$

Methods for the calculus of S. If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes,from (s_1) it results

$$S(n) = \vee S(p_i^{\alpha_i})$$

so the calculus of $S(n)$ is reduced to the calculus of $S(p^\alpha)$.

If $e_p(n)$ is the exponent of the prime p in the decomposition into primes of $n!$:

$$n! = \prod_{j=1}^{i} p_j^{e_p(n)}$$

by Legendre's formula it is said that

$$e_p(n) = \sum_{i \geq 1} \left[\frac{n}{p^i}\right]$$

Also we have

$$e_p(n) = \frac{n - \sigma_{(p)}(n)}{p-1} \tag{2}$$

where $[x]$ is the integer part of x and $\sigma_{(p)}(n)$ is the sum of digits of n in the numerical scale

$$(p) : 1\ ,\ p\ ,\ p^2\ ,\ldots,\ p^i \ldots$$

For the calculus of $S(p^\alpha)$ we need to consider in addition a generalised numerical scale $[p]$ given by:

$$[p] :\ a_1(p)\ ,\ a_2(p)\ ,\ldots,\ a_i(p)\ ,\ \ldots$$

where $a_i(p) = (p^i - 1)/(p-1)$.Then in [3] it is showed that

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \tag{3}$$

that is the value of $S(p^\alpha)$ is obtained multiplying p by the number obtained writing the exponent α in the generalised scale $[p]$ and "reading" it in the usual scale (p).

Let us observe that the calculus in the generalised scale $[p]$ is essentially different from the calculus in the scale (p).That is because if we note

$$b_n(p) = p^n$$

then for the usual scale (p) it results the recurence relation

$$b_{n+1}(p) = p \cdot b_n(p)$$

and for the generalised scale $[p]$ we have

$$a_{n+1}(p) = p \cdot a_n(p) + 1$$

For this,to add some numbers in the scale $[p]$ we do as follows:

1) We start to add from the digits of "decimals",that is from the column corresponding to $a_2(p)$.

2) If adding some digits it is obtained $pa_2(p)$,then we utilise an *unit* from the classe of "units" (the column corresponding to $a_1(p)$) to obtain $p \cdot a_2(p) + 1 = a_3(p)$.Continuing to add,if againa it is obtained $p \cdot a_2(p)$,then a new unit must be used from the classe of units,etc.

Example. If

$$m_{[5]} = 442 = 4a_3(5) + 4a_2(5) + 2a_1(5)\ ,\ n_{[5]} = 412\ ,\ r_{[5]} = 44$$

then

$$\begin{array}{rcl} m+n+r & = & 442\ + \\ & & 412 \\ & & 44 \\ & & dcba \end{array}$$

To find the digits a, b, c, d we start to add from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now,if we take an unit from the first column we get:

$$5a_2(5) + 4a_2(5) + 1 = a_3(5) + 4a_2(5)$$

so $b = 4$.

Continuing the addition we have:

$$4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$$

and using a new unit (from the first column) it results:

$$4a_3(5) + 4a_3(5) + a_3(5) + 1 = a_4(5) + 4a_3(5)$$

so $c = 4$ and $d = 1$.

Finaly,adding the remained units:

$$4a_1(5) + 2a_1(5) = 5a_1(5) + a_1(5) = 5a_1(5) + 1 = a_2(5)$$

it results that the digit $b = 4$ must be changed in $b = 5$ and $a = 0$.

So

$$m_{[5]} + n_{[5]} + r_{[5]} = 1450_{[5]} = a_4(5) + 4a_3(5) + 5a_2(5)$$

Remarque. As it is showed in [5],writing a positive integer α in the scale $[p]$ we may find the first non-zero digit on the right equals to p.Of course,that is no possible in the standard scale (p).

Let us return now to the presentation of the formulae for the calculus of the Smarandache function. For this we expresse the exponent α in both the scales (p) and $[p]$:

$$\alpha_{(p)} = c_u p^u + c_{u-1} p^{u-1} + ... + c_1 p + c_0 = \sum_{i=0}^{u} c_i p^i \tag{4}$$

and

$$\alpha_{[p]} = k_v a_v(p) + k_{v-1} a_{v-1}(p) + ... + k_1 a_1(p) = \sum_{j=1}^{v} k_j a_j(p) =$$

$$= \sum_{j=1}^{v} k_j \frac{p^j - 1}{p-1}$$

It results

$$(p-1)\alpha = \sum_{j=1}^{v} k_j p^j - \sum_{j=1}^{v} k_j \tag{5}$$

so,because $\sum_{j=1}^{v} k_j p^j = p(\alpha_{[p]})_{(p)}$, we get:

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{6}$$

From (4) we deduce

$$p\alpha = \sum_{i=0}^{u} c_i(p^{i+1} - 1) + \sum_{i=0}^{u} c_i$$

and

$$\frac{p}{p-1}\alpha = \sum_{i=0}^{u} c_i a_{i+1}(p) + \frac{1}{p-1}\sigma_{(p)}(\alpha)$$

Consequently

$$\alpha = \frac{p-1}{p}(\alpha_{(p)})_{[p]} + \frac{1}{p}\sigma_{(p)}(\alpha) \tag{7}$$

Replacing this expression of α in (6) we get:

$$S(p^\alpha) = \frac{(p-1)^2}{p}(\alpha_{(p)})_{[p]} + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \tag{8}$$

*Example.*To find $S(3^{89})$ we shall utilise the equality (3).For this we have:

$$(3) : 1 , 3 , 9 , 27 , 81 , \ldots$$
$$[3] : 1 , 4 , 13 , 40 , 121 , \ldots$$

and $89_{[3]} = 2021$,so $S(3^{89}) = 3(2021)_{(3)} = 183$.That is 183! is divisible by 3^{89} and it is the smallest factorial with this property.

We shall use now the equality (6) to calculate the same value $S(3^{89})$.For this we observe that $\sigma_{[3]}(89) = 5$ and, so $S(3^{89}) = 2 \cdot 89 + 5 = 183$.

Using (8) we get $89_{(3)} = 10022$ and :

$$S(3^{89}) = \frac{4}{3}(10022)_{[3]} + \frac{2}{3} \cdot 5 + 5 = 183$$

It is possible to expresse $S(p^\alpha)$ by mins of the exponent $e_p(\alpha)$ in the following way: from (2) and (7) it results

$$e_p(\alpha) = (\alpha_{(p)})_{[p]} - \alpha \tag{9}$$

and then from (8) and (9) it results

$$S(p^\alpha) = \frac{(p-1)^2}{p}(e_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \tag{10}$$

*Remarque.*From (3) and (8) we deduce a connection between the integer α writen in the scale $[p]$ and readed in the scale (p) and the same integer writed in the scale (p) and readed in the scale $[p]$.Namely

$$p^2(\alpha_{[p]})_{(p)} - (p-1)^2(\alpha_{(p)})_{[p]} = p\sigma_{[p]}(\alpha) + (p-1)\sigma_{(p)}(\alpha) \tag{11}$$

The function $i_p(\alpha)$. In the followings let we note $S(p^\alpha) = S_p(\alpha)$.Then from Legendre's formula it results:

$$(p-1)\alpha < S_p(\alpha) \leq p\alpha$$

that is $S(p^\alpha) = (p-1)\alpha + x = p\alpha - y$.

From (6) it results that $x = \sigma_{[p]}(\alpha)$ and to find y let us write $S_p(\alpha)$ under the forme

$$S_p(\alpha) = p(\alpha - i_p(\alpha)) \tag{12}$$

As it is showed in [4] we have $0 \leq i_p(\alpha) \leq [\frac{\alpha-1}{p}]$.

Then it results that for each function S_p there exists a function i_p so that we have the linear combination

$$\frac{1}{p}S_p(\alpha) + i_p(\alpha) = \alpha \tag{13}$$

In [1] it is proved that

$$i_p(\alpha) = \frac{\alpha - \sigma_{[p]}(\alpha)}{p} \tag{14}$$

and so it is an evident analogy between the expression of $e_p(\alpha)$ given by the equality (2) and the expression of $i_p(\alpha)$ in (14).

In [1] it is also showed that

$$\alpha = (\alpha_{[p]})_{(p)} + [\frac{\alpha}{p}] - [\frac{\sigma_{[p]}(\alpha)}{p}] = (\alpha_{[p]})_{(p)} + \frac{\alpha - \sigma_{[p]}(\alpha)}{p}$$

and so

$$S(p^\alpha) = p(\alpha - [\frac{\alpha}{p}] + [\frac{\sigma_{[p]}(\alpha)}{p}]) \tag{15}$$

Finaly,let us obserye that from the definition of Smarandache function it results that

$$(S_p \circ e_p)(\alpha) = p[\frac{\alpha}{p}] = \alpha - \alpha_p$$

where α_p is the remainder of α modulus p.Also we have

$$(e_p \circ S_p)(\alpha) \geq \alpha \text{ and } e_p(S_p(\alpha) - 1) < \alpha$$

so using (2) it results

$$\frac{S_p(\alpha) - \sigma_{(p)}(S_p(\alpha))}{p-1} \geq \alpha \text{ and } \frac{S_p(\alpha) - 1 - \sigma_{(p)}(S_p(\alpha) - 1)}{p-1} < \alpha$$

Using (6) we obtains that $S(p^\alpha)$ is the unique solution of the system

$$\sigma_{(p)}(x) \leq \sigma_{[p]}(\alpha) \leq \sigma_{(p)}(x-1)+1$$

The calculus of card$(S^{-1}(n))$.Let $q_1, q_2, ..., q_h$ be all the prime itegers smallest then n and non dividing n. Let also denote shortly $e_{q_j}(n) = f_j$.A solution x_0 of the equation

$$S(x) = n$$

has the property that x_0 divides $n!$ and non divides $(n-1)!$.Now, if $d(n)$ is the number of positive divisors of n,from the inclusion

$$\{m \ / \ m \text{ divides } (n-1)!\} \subset \{m \ / \ m \text{ divides } n!\}$$

and using the definition of Smarandache function it results that

$$\text{card}(S^{-1}(n)) = d(n!) - d((n-1)!) \tag{16}$$

Example. In [6] A. Stuparu and D. W. Sharpe has proved that if p is a given prime,the equation

$$S(x) = p$$

has just $d((p-1)!)$ solutions (all of them in between p and $p!$) .Let us observe that $e_p(p!) = 1$ and $e_p((p-1)!) = 0$,so because

$$d(p!) = (e_p(p!)+1)(f_1+1)(f_2+1)...(f_h+1) = 2(f_1+1)(f_2+1)...(f_h+1)$$
$$d((p-1)!) = (f_1+1)(f_2+1)...(f_h+1)$$

it results

$$\text{card}(S^{-1}(p!)) = d(p!) - d((p-1)! = d((p-1)!)$$

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THE FIRST CONSTANT OF SMARANDACHE

by

Ion Cojocaru and Sorin Cojocaru

In this note we prove that the series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a real number $s \in (0.717, 1.253)$ that we call the first constant constant of Smarandache.

It appears as an open problem, in [1], the study of the nature of the series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$. We can write it as it follows :

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{3!} + \cdots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \cdots =$$

$$= \sum_{n=2}^{\infty} \frac{a(n)}{n!}, \text{ where } a(n) \text{ is the number of the equation } S(x) = n, n \in \mathbf{N}, n \geq 2 \text{ solutions.}$$

It results from the equality $S(x) = n$ that x is a divisor of n!, so a(n) is smaller than d(n!).

So, $\quad a(n) < d(n!). \qquad (1)$

Lemma 1. We have the inequality :

$$d(n) \leq n - 2, \text{ for each } n \in \mathbf{N}, n \geq 7. \qquad (2)$$

Proof. Be $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with $p_1, p_2, ..., p_k$ prime numbers, and $a_i \geq 1$ for each $i \in \in \{1, 2, ..., k\}$. We consider the function $f : [1, \infty) \to \mathbf{R}$, $f(x) = a^x - x - 2$, $a \geq 2$, fixed. It is derivable on $[1, \infty)$ and $f'(x) = a^x \ln a - 1$. Because $a \geq 2$, and $x \geq 1$ it results that $a^x \geq 2$, so $a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1$, i.e., $f'(x) > 0$ for each $x \in [1, \infty)$ and $a \geq 2$, fixed. But $f(1) = a - 3$. It results that for $a \geq 3$ we have $f(x) \geq 0$, that means $a^x \geq x + 2$.

Particularly, for $a = p_i$, $i \in \{1, 2, ..., k\}$, we obtain $p_i^{a_i} \geq a_i + 2$ for each $p_i \geq 3$.

If $n = 2^s$, $s \in \mathbf{N}^*$, then $d(n) = s + 1 < 2^s - 2 = n - 2$ for $s \geq 3$.

So we can assume $k \geq 2$, i.e. $p_2 \geq 3$. It results the inequalities :

$$p_1^{a_1} \geq a_1 + 1$$

$$p_2^{a_2} \geq a_2 + 2$$

.....

$$p_k^{a_k} \geq a_k + 2,$$

equivalent with

$$p_1^{a_1} \geq a_1 + 1,\ p_2^{a_2} - 1 \geq a_2 + 1,\ \ldots,\ p_k^{a_k} - 1 \geq a_k + 1. \tag{3}$$

Multiplying, member with member, the inequalities (3) we obtain :

$$p_1^{a_1}(p_2^{a_2} - 1)\cdots(p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1)\cdots(a_k + 1) = d(n). \tag{4}$$

Considering the obvious inequality :

$$n - 2 \geq p_1^{a_1}(p_2^{a_2} - 1)\cdots(p_k^{a_k} - 1) \tag{5}$$

and using (4) it results that :

$n - 2 \geq d(n)$ for each $n \geq 7$.

Lemma 2. $d(n!) < (n - 2)!$ for each $n \in \mathbf{N}, n \geq 7$. (6)

Proof. We ration trough induction after n. So, for $n = 7$,

$d(7!) = d(2^4 \cdot 3^2 \cdot 5 \cdot 7) = 60 < 120 = 5!$.

We assume that $d(n!) < (n - 2)!$.

$d((n+1)!) = d(n!(n + 1)) \leq d(n!) \cdot d(n + 1) < (n - 2)!\ d(n + 1) < (n-2)!\ (n - 1) = (n - 1)!$,

because in accordance with Lemma 1, $d(n + 1) < n - 1$.

Proposition. The series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a number $s \in (0.717, 1.253)$, that we call the first constant constant of Smarandache.

Proof. From Lemma 2 it results that $a(n) < (n-2)!$, so $\frac{a(n)}{n!} < \frac{1}{n(n-1)}$ for every $n \in \mathbf{N}$, $n \geq 7$ and $\sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^{6} \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)}$.

Therefore $\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^2-n}$. (7)

Because $\sum_{n=2}^{\infty} \frac{1}{n^2-n} = 1$ we have : it exists the number $s > 0$, that we call the Smarandache constant, $s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}$.

From (7) we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^2-2} - \frac{1}{3^2-3} - \frac{1}{4^2-4} +$$

$$+ \frac{1}{5^2-5} + \frac{1}{6^2-6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1,253.$$

But, because $S(n) \leq n$ for every $n \in \mathbf{N}^*$, it results :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

Consequently, for this first constant we obtain the framing $e - 2 < s < 1,253$, i.e., $0,717 < s < 1,253$.

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DEPARTMENT OF MATHEMATICS

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THE SECOND CONSTANT OF SMARANDACHE

by

Ion Cojocaru and Sorin Cojocaru

In the present note we prove that the sum of remarcable series $\sum_{n\geq 2} \frac{S(n)}{n!}$, which implies the Smarandache function is an `irrational number(second constant of Smarandache)`.

Because $S(n) \leq n$, it results $\sum_{n\geq 2} \frac{S(n)}{n!} \leq \sum_{n\geq 2} \frac{1}{(n-1)!}$. Therefore the serie $\sum_{n\geq 2} \frac{S(n)}{n!}$ is convergent to a number f.

Proposition. The sum f of the series $\sum_{n\geq 2} \frac{S(n)}{n!}$ is an irrational number.

Proof. From the precedent lines it results that $\lim_{n\to\infty} \sum_{i=2}^{n} \frac{S(n)}{n!} = f$. Against all reson we assume that $f \in Q$, $f > 0$. Therefore it exists $a,b \in N$, $(a,b) = 1$, so that $f = \frac{a}{b}$.

Let p be a fixed prime number, $p > b$, $p \geq 3$. Obviously, $\frac{a}{b} = \sum_{i=2}^{p-1} \frac{S(i)}{i!} + \sum_{i\geq p} \frac{S(i)}{i!}$ which leads to:

$$\frac{(p-1)!a}{b} = \sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} + \sum_{i\geq p} \frac{(p-1)!S(i)}{i!}$$

Because $p > b$ it results that $\frac{(p-1)!a}{b} \in N$ and $\sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} \in N$. Consequently we have $\sum_{i\geq p} \frac{(p-1)!S(i)}{i!} \in N$ too.

Be $\alpha = \sum_{i\geq p} \frac{(p-1)!S(i)}{i!} \in N$. So we have the relation

$$\alpha = \frac{(p-1)!S(p)}{p!} + \frac{(p-1)!S(p+1)}{(p+1)!} - \frac{(p-1)!S(p+2)}{(p+2)!} + \dots .$$

Because p is a prime number it results $S(p) = p$.

So

$$\alpha = 1 + \frac{S(p+1)}{p(p+1)} + \frac{S(p+2)}{p(p+1)(p+2)} + \dots > 1 \qquad (1)$$

We know that $S(p+i) \leq p+i$ $(\forall) i \geq 1$, with equality only if the number $p+i$ is prime. Consequently, we have

$$\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)} + \ldots < 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \ldots = \frac{p}{p-1} < 2 \qquad (2)$$

From the inequalities (1) and (2) it results that $1 < \alpha < 2$, impossible, because $\alpha \in N$. The proposition is proved.

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THE THIRD AND FOURTH CONSTANTS OF SMARANDACHE

by

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In the present note we prove the divergence of some series involving the Smarandache function, using an unitary method, and then we prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)....S(n)}$$

is convergent to a number $s \in (71/100, 101/100)$ and we study some applications of this series in the Number Theory (third constant of Smarandache).

The Smarandache Function $S : N^* \to N$ is defined [1] such that $S(n)$ is the smallest integer k with the property that k! is divisible by n.

Proposition 1. If $(x_n)_{n \geq 1}$ is a strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \tag{1}$$

where S is the Smarandache function, is divergent.

Proof. We consider the function $f : [x_n, x_{n+1}] \to R$, defined by $f(x) = \ln \ln x$. It fulfils the conditions of the Lagrange's theorem of finite increases. Therefore there is $c_n \in (x_n, x_{n+1})$ such that :

$$\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n} (x_{n+1} - x_n). \tag{2}$$

Because $x_n < c_n < x_{n+1}$, we have :

$$\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, \quad (\forall) n \in N, \tag{3}$$

if $x_n \neq 1$.

We know that for each $n \in N^* \setminus \{1\}$, $\frac{S(n)}{n} \leq 1$, i.e.

$$0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}, \tag{4}$$

from where it results that $\lim_{n \to \infty} \frac{S(n)}{n \ln n} = 0$. Hence there exists $k > 0$ such that $\frac{S(n)}{n \ln n} < k$, i.e., $n \ln n > \frac{S(n)}{k}$ for any $n \in N^*$, so

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \tag{5}$$

Introducing (5) in (3) we obtain :

$$\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}, \ (\forall) n \in N^* \setminus \{1\}. \tag{6}$$

Summing up after n it results :

$$\sum_{n=1}^{m} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k} (\ln \ln x_{m+1} - \ln \ln x_1).$$

Because $\lim_{m \to \infty} x_m = \infty$ we have $\lim_{m \to \infty} \ln \ln x_m = \infty$, i.e., the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

Proposition 2. Series $\sum_{n=2}^{\infty} \frac{1}{S(n)}$, where S is the Smarandache function, is divergent.

Proof. We use Proposition 1 for $x_n = n$.

Remarks. 1) If x_n is the n - th prime number, then the series $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$ is divergent.

2) If the sequence $(x_n)_{n \geq 1}$ forms an arithmetical progression of natural numbers, then the series $\sum_{n=1}^{\infty} \frac{1}{S(x_n)}$ is divergent.

3) The series $\sum_{n=1}^{\infty} \frac{1}{S(2n+1)}$, $\sum_{n=1}^{\infty} \frac{1}{S(4n+1)}$ etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent aspects are divergent.

Proposition 3. The series :

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)},$$

where S is the Smarandache function, is convergent to a number $s \in (71/100, 101/100)$.

Proof. From the definition of the Smarandache function it results $S(n) \leq n!$, $(\forall)n \in \mathbf{N}^* \setminus \{1\}$, so $\frac{1}{S(n)} \geq \frac{1}{n!}$.

Summing up, begining with $n = 2$ we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

The product $S(2) \cdot S(3) \ldots S(n)$ is greater than the product of prime numbers from the set $\{1, 2, \ldots, n\}$, because $S(p) = p$, for p = prime number. Therefore :

$$\frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=1}^{k} p_i}, \tag{7}$$

where p_k is the biggest prime number smaller or equal to n.

There are the inequalities :

$$S = \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \cdots +$$

$$+ \frac{1}{S(2)S(3)\cdots S(k)} + \cdots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{4}{2 \cdot 3 \cdot 5 \cdot 7} +$$

$$+ \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \cdots + \frac{p_{k+1} - p_k}{p_1 p_2 \cdots p_k} + \cdots. \tag{8}$$

Using the inequality $p_1 p_2 \cdots p_k > p_{k+1}^3$, $(\forall)k \geq 5$ [2], we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots + \frac{1}{p_{k+1}^2} + \cdots . \tag{9}$$

We note $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots$ and observe that $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \cdots$.

It results :

$$P < \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{12^2}\right),$$

where

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \text{ (EULER).}$$

Introducing in (9) we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots - \frac{1}{12^2}.$$

Estimating with an approximation of an order not more than $\frac{1}{10^2}$, we find :

$$0,71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 1,01. \tag{10}$$

The Proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right framing :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 0,97. \tag{11}$$

Proposition 4. Let α be a fixed real number, $\alpha \geq 1$. Then the series $\sum_{n=2}^{\infty} \frac{n^{\alpha}}{S(2)S(3)\cdots S(n)}$ is convergent (fourth constant of Smarandache).

Proof. Be $(p_k)_{k \geq 1}$ the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1}$$

$$\frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1p_2}$$

$$\frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1p_2} < \frac{p_3^\alpha}{p_1p_2}$$

$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}$$

$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}$$

.....

$$\frac{n^\alpha}{S(2)S(3)\cdots S(n)} < \frac{n^\alpha}{p_1p_2\cdots p_k} < \frac{p_{k+1}^\alpha}{p_1p_2\cdots p_k},$$

where $p_i \le n$, $i \in \{1, ..., k\}$, $p_{k+1} > n$.

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{(p_{k+1} - p_k)\cdot p_{k+1}^\alpha}{p_1p_2\cdots p_k} <$$

$$< 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{p_{k+1}^{\alpha+1}}{p_1p_2\cdots p_k}.$$

Then it exists $k_0 \in \mathbf{N}$ such that for any $k \ge k_0$ we have :

$$p_1p_2\cdots p_k > p_{k+1}^{\alpha+3}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1p_2\cdots p_k} + \sum_{k\ge k_0} \frac{1}{p_{k+1}^2}$$

Because the series $\sum_{k \geq k_0} \frac{1}{p_{k+1}^2}$ is convergent it results that the given series is convergent too.

Consequence 1. It exists $n_0 \in \mathbf{N}$ so that for each $n \geq n_0$ we have $S(2)S(3) \ldots S(n) > n^{\alpha}$.

Proof. Because $\lim_{n \to \infty} \frac{n^{\alpha}}{S(2)S(3) \cdots S(n)} = 0$, there is $n_0 \in \mathbf{N}$ so that

$$\frac{n^{\alpha}}{S(2)S(3) \cdots S(n)} < 1 \text{ for each } n \geq n_0.$$

Consequence 2. It exists $n_0 \in \mathbf{N}$ so that :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \cdot n^{\frac{\alpha}{n-1}} \text{ for each } n \geq n_0.$$

Proof. We apply the inequality of averages to the numbers S(2), S(3), ..., S(n) :

$$S(2) + S(3) + \cdots + S(n) > (n-1)\sqrt[n-1]{S(2)S(3) \cdots S(n)} > (n-1)n^{\frac{\alpha}{n-1}}, \quad \forall n \geq n_0.$$

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DEPARTMENT OF MATHEMATICS

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A PARADOXICAL MATHEMATICIAN: HIS FUNCTION, PARADOXIST GEOMETRY, AND CLASS OF PARADOXES

by Michael R. Mudge

Described by Charles T. Le, Bulletin of Number Theory, vol.3, No.1, March 1995,as "The most paradoxist mathematician of the world" FLORENTIN SMARANDACHE was born on December 10th, 1954, in Balcesti (a large village), Valcea, Romania of peasant stock. A very hard and socially deprived childhood led to a period of eccentric teenage behaviour, he was close to being expelled from his high school in Craiova for disciplinary reasons. Eventually, however, a period of university studies, 1974-79, resulted in the recognition of mathematical brilliance by the professor of algebra, Alexandru Dinca. Florentin generalised Euler's Theorem from:

If $(a,m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$ to,

If $(a,m) = d_s$, then $a^{\phi(m_s + s)} \equiv a^s \pmod{m}$ where m_s divides m and s is the number of steps to get m_s.

An industrial appointment from 1979 to 1981 was disastrous, ending in dismissal for disciplinary reasons. In 1986 an apparently successful teaching appointment was terminated by the Ceausescu dictatorship and two years of unemployment followed. In 1988 an illegal escape from Romania through Bulgaria resulted in two years in a Turkish refugee camp, where much time was spent as a drunken vagrant.

Many mathematicians and writers lobbied the United Nations Commision for Refugees,based in Rome, and exile to the United States followed in 1990. As a member of the American Mathematical Society since 1992 and of the Romanian Scientists Association (Bucharest) since 1993 and a reviewer for the Number Theory to Zentralblatt für Mathematik scores of publications and four books bear the name of Smarandache, publishing in English, French and his native Romanian.

The Smarandache Function, S(n), is defined for positive integer argument only as the smallest integer such that S(n)! is divisible by n: (the extension to other real/complex argument has not, yet, been investigated).

The Smarandache Quotient, Q(n), is defined to be S(n)!/n.

Limited tabulation of both functions appears in The Encyclopedia of Integer Sequences by N.J.A.Sloane & Simon Plouffe, Academic Press, 1995. M1669 & M0453.

There exists an extensive literature dealing with properties of these functions; "An Infinity of Unsolved Problems Concerning a Function in Number Theory", Smarandache Function Journal, vol.1., No.1, December 1990, pp12 - 55, ISSN 1053-4792, Number Theory Publishing Company, P.O. Box 42561, Phoenix, Arizona 85080, USA providing an ideal starting point for interested readers.

A recent paper by Charles Ashbacher, Mathematical Spectrum, 1995/96, vol.28., No.1, pp20-21 addresses the question of when the Smarandache Function satisfies a Fibonacci recurrence relation, i.e. S(n) = S(n-1) + S(n-2). Empirical evidence is for 'few' occasions the largest known being n = 415664. Are there infitely many?

A Paradoxist Geometry. In 1969, at the age of 15, fascinated by geometry, Florentin Smarandache constructed a partially Euclidean and partially non-Euclidean geometry in the same space by a strange replacement of the Euclid's fifth postulate (the axiom of paralells) with the following five-statement proposition:

a)there are at least a straight line and an exterior point to it in this space for which only one line passes through the point and does not intersect the initial line;

b)there are at least a straight line and an exterior point to it in this space for which only a finite number of lines, say $k \geqslant 2$, pass through the point and do not intersect the initial line;

c)there are at least a straight line and an exterior point to it in this space for which any line that passes through the point intersects the initial line;

d)there are at least a straight line and an exterior point to it

in this space for which an infinite number of lines that pass through the point (but not all of them) do not intersect the initial line;

e)there are at least a straight line and an exterior point to it in this space for which any line that passess through the point does not intersect the initial line.

Does there exist a model for this PARADOXIST GEOMETRY? If not can a contradiction be found using the above set of propositions together with Euclid's remaining Axioms?

Smarandache Classes of Paradoxes. Contributed by Dr.Charles T.Le, Erhus University, Box 10163, Glendale, ARIZONA 85318. USA.

Let @ be an attribute and non-@ its negation.

Thus if @ means 'possible' then non-@ means 'impossible'.

The original set of Smarandache Paradoxes are:

ALL is "@", THE "NON-@" TOO.

ALL IS "NON-@", THE "@" TOO.

NOTHING IS "@", NOT EVEN "@".

These three kinds of paradox are mutually equivalent and reduce to:

PARADOX: ALL (verb) "@", THE "NON-@" TOO.

See Florentin Smarandache, "Mathematical Fancies & Paradoxes", paper presented at the Eugene Strens Memorial on Intuitive and Recreational Mathematics and its History, University of Calgary, Alberta, Canada, July 27 - August 2, 1986.

8/10/95

Further Reading:

Only Problems, Not Solutions!, Florentin Smarandache, Xiquan Publishing House, 1993 (fourth edition), ISBN- 1-879585-00-6.

Some Notions and Questions in Number Theory, C.Dumitrescu & V.Seleacu, Ehrus University Press, Glendale,1994, ISBN 1-879585-48-0.

Smarandache - Fibonacci Triplets

H. Ibstedt

We recall the definition of the Smarandache Function S(n):

S(n) = the smallest positive integer such that S(n)! is divisible by n.

and the Fibonacci recurrence formula:

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

which for $F_0 = F_1 = 1$ defines the Fibonacci series.

This article is concerned with isolated occurrencies of triplets n, n-1, n-2 for which S(n)=S(n-1)+S(n-2). Are there infinitly many such triplets? Is there a method of finding such triplets that would indicate that there are in fact infinitely many of them.

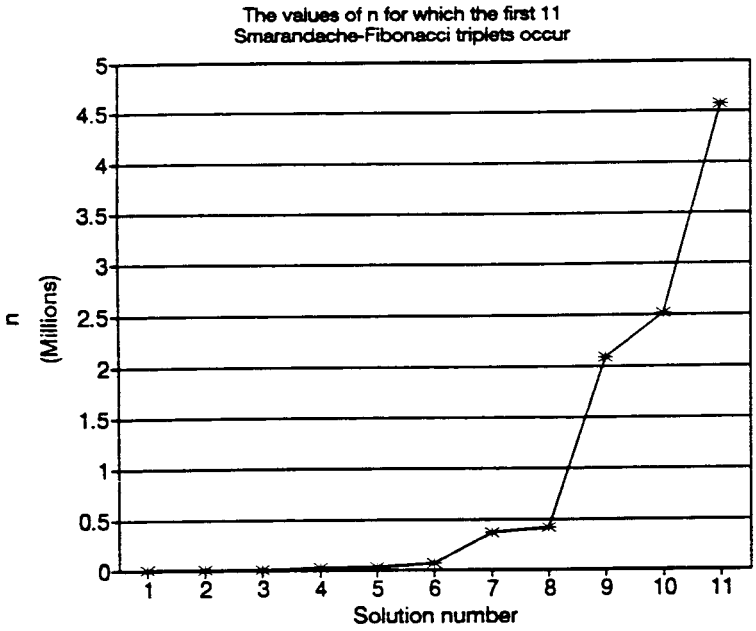
A straight forward search by applying the definition of the Smarandache Function to consecutive integers was used to identify the first eleven triplets which are listed in table 1. As often in empirical number theory this merely scratched the surface of the ocean of integers. As can be seen from diagram 1 the next triplet may occur for a value of n so large that a sequential search may be impractical and will not make us much wiser.

Table 1. The first 11 Smarandache-Fibonacci Triplets

#	n	S(n)	S(n-1)	S(n-2)
1	11	11	5	2*3
2	121	2*11	5	17
3	4902	43	29	2*7
4	26245	181	18	163
5	32112	223	197	2*13
6	64010	173	2*23	127
7	368140	233	2*41	151
8	415664	313	2*73	167
9	2091206	269	2*101	67
10	2519648	1109	2*101	907
11	4573053	569	2*53	463

However, an interesting observation can be made from the triplets already found. Apart from n=26245 the Smarandache-Fibonacci triplets have in common that one member is two times a prime number while the other two members are prime numbers. This observation

Diagram1.



leads to a method to search for Smarandache-Fibonacci triplets in which the following two theorems play a rôle:

I. If $n=ab$ with $(a,b)=1$ and $S(a)<S(b)$ then $S(n)=S(b)$.

II. If $n = p^a$ where p is a prime number and $a \leq p$ then $S(p^a)=ap$.

The search for Smarandache-Fibonacci triplets will be restricted to integers which meet the following requirements:

$$n = xp^a \text{ with } a \leq p \text{ and } S(x)<ap \tag{1}$$

$$n-1 = yq^b \text{ with } b \leq q \text{ and } S(y)<bq \tag{2}$$

$$n-2 = zr^c \text{ with } c \leq r \text{ and } S(z)<cr \tag{3}$$

p,q and r are primes. We then have $S(n)=ap$, $S(n-1)=bq$ and $S(n-2)=cr$. From this and by subtracting (2) from (1) and (3) from (2) we get

$$ap = bq + cr \tag{4}$$

$$xp^a - yq^b = 1 \tag{5}$$

$$yq^b - zr^c = 1 \tag{6}$$

TABLE 2. Smarandache - Fibonacci Triplets.

#	N	S(N)	S(N-1)	S(N-2)	t
1	4	4 *	3	2 *	0
2	11	11	5	6 *	0
3	121	22 *	5	17	0
4	4902	43	29	14 *	-4
5	32112	223	197	26 *	-1
6	64010	173	46 *	127	-1
7	368140	233	82 *	151	-1
8	415664	313	167	146 *	-8
9	2091206	269	202 *	67	-1
10	2519648	1109	202 *	907	0
11	4573053	569	106 *	463	-3
12	7783364	2591	202 *	2389	0
13	79269727	2861	2719	142 *	10
14	136193976	3433	554 *	2879	-1
15	321022289	7589	178 *	7411	5
16	445810543	1714 *	761	953	-1
17	559199345	1129	662 *	467	-5
18	670994143	6491	838 *	5653	-1
19	836250239	9859	482 *	9377	1
20	893950202	2213	2062 *	151	0
21	937203749	10501	10223	278 *	-9
22	1041478032	2647	1286 *	1361	-1
23	1148788154	2467	746 *	1721	3
24	1305978672	5653	1514 *	4139	0
25	1834527185	3671	634 *	3037	-5
26	2390706171	6661	2642 *	4019	0
27	2502250627	2861	2578 *	283	-1
28	3969415464	5801	1198 *	4603	-2
29	3970638169	2066 *	643	1423	-6
30	4652535626	3506 *	3307	199	0
31	6079276799	3394 *	2837	557	-1
32	6493607750	3049	1262 *	1787	5
33	6964546435	2161	1814 *	347	-4
34	11329931930	3023	2026 *	997	-4
35	11695098243	12821	1294 *	11527	2
36	11777879792	2174 *	1597	577	6
37	13429326313	4778 *	1597	3181	1
38	13849559620	6883	2474 *	4409	1
39	14298230970	2038 *	1847	191	7
40	14988125477	3209	2986 *	223	2
41	17560225226	4241	3118 *	1123	-2
42	18704681856	3046 *	1823	1223	4
43	23283250475	4562 *	463	4099	-10
44	25184038673	5582 *	1951	3631	-2
45	29795026777	11278 *	8819	2459	0
46	69481145903	6301	3722 *	2579	3
47	107456166733	10562 *	6043	4519	-1
48	107722646054	8222 *	6673	1549	-1
49	122311664350	20626 *	10463	10163	0
50	126460024832	6917	2578 *	4339	11
51	155205225351	8317	4034 *	4283	-5
52	196209376292	7246 *	3257	3989	-5
53	210621762776	6914 *	1567	5347	11
54	211939749997	16774 *	11273	5501	0
55	344645609138	7226 *	2803	4423	9
56	484400122414	16811	12658 *	4153	-1
57	533671822944	21089	18118 *	2971	0
58	620317662021	21929	20302 *	1627	0
59	703403257356	13147	10874 *	2273	-2
60	859525157632	14158 *	3557	10601	-5
61	898606860813	19973	13402 *	6571	1
62	972733721905	10267	10214 *	53	-4
63	1185892343342	18251	12022 *	6229	-2
64	1225392079121	12202 *	9293	2909	-4
65	1294530625810	17614 *	5807	11807	-3
66	1517767218627	11617	8318 *	3299	-8
67	1905302845042	22079	21478 *	601	-1
68	2679220490034	11402 *	7459	3943	11
69	3043063820555	14951	12202 *	2749	5
70	6098616817142	24767	20206 *	4561	2
71	6505091986039	31729	19862 *	11867	2
72	13666465868293	28099	16442 *	11657	7

The greatest common divisor $(p^a, q^b)=1$ obviously divides the right hand side of (5). This is the condition for (5) to have infinitely many solutions for each solution (p,q) to (4). These are found using Euclid's algorithm and can be written in the form:

$$x = x_0 + q^b t, \qquad y = y_0 - p^a t \tag{5'}$$

where t is an integer and (x_0,y_0) is the principal solution.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for z.

$$z = (yq^b - 1)/r^c \tag{6'}$$

Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and (a,b,c)=(1,1,2) with the parameter t restricted to the interval $-11 \le t \le 11$. The result is shown in table 2. Since the correctness of these calculations are easily verfied from factorisations of S(n), S(n-1), and S(n-2) these are given in table 3 for two large solutions taken from an extension of table 2.

Table 3. Factorisation of two Smarandache-Fibonacci Triplets.

n=	16,738,688,950,356 = $2^2 \cdot 3 \cdot 31 \cdot 193 \cdot 15{,}269^2$	S(n)=	2·15,269
n-1=	16,738,688,950,355 = $5 \cdot 197 \cdot 1{,}399 \cdot 1{,}741 \cdot 6{,}977$	S(n-1)=	6,977
n-2=	16,738,688,950,354 = $2 \cdot 7^2 \cdot 19 \cdot 23 \cdot 53 \cdot 313 \cdot 23{,}561$	S(n-2)=	23,561
n=	19,448,047,080,036 = $2^2 \cdot 3^2 \cdot 43^2 \cdot 17{,}093^2$	S(n)=	2·17,093
n-1=	19,448,047,080,035 = $5 \cdot 7 \cdot 19 \cdot 37 \cdot 61 \cdot 761 \cdot 17{,}027$	S(n-1)=	17,027
n-2=	19,448,047,080,034 = $2 \cdot 97 \cdot 1{,}609 \cdot 3{,}631 \cdot 17{,}159$	S(n-1)=	17,159

Conjecture. There are infinitely many triplets n, n-1, n-2 such that S(n)=S(n-1)+S(n-2).

Questions:

1. It is interesting to note that there are only 7 cases in table 2 where S(n-2) is two times a prime number and that they all occur for relatively small values of n. Which is the next one?

2. The solution for n=26245 stands out as a very interesting one. Is it a unique case or is it a member of family of Smarandache-Fibonacci triplets different from those studied in this article?

References:

C. Ashbacher and M. Mudge, *Personal Computer World*, October 1995, page 302.

M. Mudge, in a Letter to R. Muller (05/14/96), states that:
"John Humphries of Hulse Ground Farm, Little Faringdo, Lechlade, Glovcester, GL7 3QR, U.K., has found a set of three numbers, greater than 415662, whose Smarandache Function satisfies the Fibonacci Recurrence, i.e. S(2091204) = 67, S(2091205) = 202, S(2091206) = 269, and 67 + 202 = 269.

THE SOLUTION OF SOME DIOPHANTINE EQUATIONS RELATED TO SMARANDACHE FUNCTION

by

Ion Cojocaru and Sorin Cojocaru

In the present note wesolve two diophantine eqations concerning the Smarandache function.

First, we try to solve the diophantine eqation :

$$S(x^y) = y^x \qquad (1)$$

It is porposed as an open problem by F. Smarandache in the work [1], pp. 38 (the problem 2087).

Because S(1) = 0, the couple (1,0) is a solution of eqation (1). If x = 1 and y ≥1, the eqation there are no (1,y) solutions. So, we can assume that x ≥ 2. It is obvious that the couple (2,2) is a solution for the eqation (1).

If we fix y = 2 we obtain the equation $S(x^2) = 2^x$. It is easy to verify that this eqation has no solution for $x \in \{3,4\}$, and for x ≥5 we have $2^x > x^2 \geq S(x^2)$, so $2^x > S(x^2)$. Consequently for every $x \in N^* \setminus \{2\}$, the couple (x,2) isn't a solution for the eqation (1).

We obtein the equation $S(2^y) = y^2$, y ≥3, fixing x = 2. It is know that for p = prime number we have the ineqality:

$$S(p^r) \leq p \cdot r \qquad (2)$$

Using the inequality (2) we obtein the inequality $S(2^y) \leq 2 \cdot y$. Because y ≥3 implies $y^2 > 2y$, it results $y^2 > S(2^y)$ and we can assume that x ≥3 and y ≥3.

We consider the function f: $[3,\infty] \to R_+^*$ defined by $f(x) = \frac{y^x}{x^y}$, where y ≥ 3 is fixed.

This function is derivable on the considered interval, and $f'(x) = \frac{y^x x^{y-1}(x \ln y - y)}{x^{2y}}$. In the point $x_o = \frac{y}{\ln y}$ it is equal to zero, and $f(x_o) = f(\frac{y}{\ln y}) = y^{\frac{1}{\ln y}} (\ln y)^y$.

Because y ≥ 3 it results that lny > 1 and $y^{\frac{1}{\ln y}} > 1$, so $f(x_o) > 1$. For $x > x_o$, the function f is strict incresing, so $f(x) > f(x_o) > 1$, that leads to $y^x > x^y \geq S(x^y)$, respectively $y^x > S(x^y)$. For $x < x_o$, the function f is strict decreasing, so $f(x) > f(x_o) > 1$ that lands to $y^x > S(x^y)$. There fore, the only solution of the eqaution (1) are the couples (1,0) and (2,2).

SOLVING THE DIOPHANTINE EQUATION

$$x^y - y^x = S(x) \qquad (3)$$

It is obvious that the couples (1,1) is a solution of the eqaution (3).

Because $x^y - y^x = S(x)$ it results $x \neq y$ (otherwise we have S(x)=0, i.e., x = 1 = y). We prove that the equation (3) has an unique solution.

Case I : $x > y$. Therefore it exists $a \in N^*$ so $x = y + a$, $(y+a)^y - y^{y+a} = S(y+a)$ or $(1+\frac{a}{y})^y - y^a = \frac{S(y+a)}{y^y}$. But $(1+\frac{a}{y})^y < e^a$. It results $e^a - y^a > \frac{S(y+a)}{y^y}$, false inequality for $y > e$ ($e^a - y^a < 0$ for $y > e$). So we have y = 1 or y = 2. If y = 1 we have x-1 = S(x). In this situation it is obvious that x is a compound number. If $x = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ is the factorization of x into prims wich $p_i \neq p_j$, $a_i \neq 0$, $i,j = \overline{1,n}$, then we have $S(x) = \max_{1 \leq s \leq n} S(p_i^{a_i}) = S(p_e^{a_e})$, $1 \leq e \leq r$. But, because $S(x) = S(p_e^{a_e}) < p_e a_e < x - 1$ it results that $S(x) < x - 1$, that is fals.

If y = 2, we have $x^2 - 2^x = S(x)$. For $x \geq 4$ we obtein $x^2 - 2^x < 0$, and for $x \in \{2,3\}$ there is no solution.

Case II: $x < y$. Therefore it exists $b > 0$ such that $y = x + b$. Then we have $x^{x+b} - (x+b)x = S(x)$, so $x^b - (1+\frac{b}{x})^x = \frac{S(x)}{x^x} \leq \frac{x}{x^x} \leq 1$.

But, because $(1+\frac{b}{x})^x < e^b$ we obtain $x^b - e^b < 1$, which is a false inequality for $x \geq 4$. If x = 2, then $2^y - y^2 = 2$, an equation which fas no solution because $2^y - y^2 \geq 7$ for $y \geq 5$.

If x = 3, then $3^y - y^3 = 3$, an equation which has no solutions for $y \in \{1,2,3\}$, because, if $y \geq 4$ it results $3^y - y^3 \geq 17$.

Therefore the equation (3) admits an unique solution (1,1).

REFERENCES

[1] **F. Smarandache** : *An infinity of unsolved problems concerning a Function in the Number Theory* (Presented at the 14th American Romanian Academy Anual cOnvention, hold in Los Angeles, California, hosted by the University of Southern California, from April 20 to April 22, 1989).

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Problems

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Welcome to the inaugural version of what is to be a regular feature in **Smarandache Notions**! Our goal is to present interesting and challenging problems in all areas and at all levels of difficulty with the only limits being good taste. Readers are encouraged to submit new problems and solutions to the editor at one of the addresses given above. All solvers will be acknowledged in a future issue. Please submit a solution along with your proposals if you have one. If there is no solution and the editor deems it appropriate, that problem may appear in the companion column of unsolved problems. Feel free to submit computer related problems and use computers in your work. Programs can also be submitted as part of the solution. While the editor is fluent in many programming languages, be cautious in submitting programs as solutions. Wading through several pages of obtuse program to determine if the submitter has done it right is not the editors idea of a good time. Make sure you explain things in detail.

If no solution is currently available, the problem will be flagged with an asterisk*. The deadline for submission of solutions will generally be six months after the date appearing on that issue. Regardless of deadline, no problem is ever officially closed in the sense that new insights or approaches are always welcome. If you submit a problem or solution and wish to guarantee a reply, please include a self-addressed envelope or postcard with appropriate stamps attached. Suggestions for improvement or modification are also welcome at any time. All proposals in this initial offering are by the editor.

The Smarandache function S(n) is defined in the following way

For $n \geq 1$, $S(n) = m$ is the smallest nonnegative integer such that n evenly divides m factorial.

New Problems

1) The Euler phi function $\phi(n)$ is defined to the number of positive integers not exceeding n that are relatively prime to n.

a) Prove that there are no solutions to the equation

$$\phi(S(n)) = n$$

b) Prove that there are no solutions to the equation

$$S(\phi(n)) = n$$

c) Prove that there are an infinite number of solutions to the equation

$$n - \phi(S(n)) = 1$$

d) Prove that for every odd prime p, there is a number n such that

$$n - \phi(S(n)) = p+1$$

2) This problem was proposed in **Canadian Mathematical Bulletin** by P. Erdos and was listed as unsolved in the book **Index to Mathematical Problems 1980-1984** edited by Stanley Rabinowitz and published by MathPro Press.

Prove that for infinitely many n

$$\phi(n) < \phi(n - \phi(n)).$$

3) The following appeared as unsolved problem (21) in **Unsolved Problems Related To Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Are there m,n,k non-null positive integers, $m,n \neq 1$ for which

$$S(mn) = m^k * S(n)?$$

Find a solution.

4) The following appeared as unsolved problem (22) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Is it possible to find two distinct numbers k and n such that

$$\log_{S(k^n)} S(n^k)$$

is an integer?

Find two integers n and k that satisfy these conditions.

5) Solve the following doubly true Russian alphametic

```
  ДВА        2
  ДВА        2
  ТРИ        3
 -------    --
 СЕМЬ        7
```

where 2 divides ДВА, 3 divides ТРИ and 7 divides СЕМЬ.

Can anyone come up with a similar Romanian alphametic?

6) Prove the Smarandache Divisibility Theorem

If a and m are integers and $m > 0$, then

$(a^m - a)(m - 1)!$

is divisible by m.

Which was problem (126) in **Some Notions and Questions in Number Theory**, published by Erhus University Press.

7) Let D = { 0,1,2,3,4,5,6,7,8,9 }. For any number $1 \leq n \leq 10$, we can take n unique digits from D and form a number, leading zero not allowed. Let P_n be the set of all numbers that can be formed by choosing n unique digits from D. If 1 is not considered prime, which of the sets P_n contains the largest percentage of primes?

This problem is similar to unsolved problem 3 part (a) that appeared in **Only Problems, Not Solutions**, by Florentin Smarandache.

*8) The following four problems are all motivated by unsolved problem 3 part (b) that appeared in **Only Problems, Not Solutions**, by Florentin Smarandache.

a) Find the smallest integer n such that n! contains all 10 decimal digits.

b) Find the smallest integer n such that the n-th prime contains all 10 decimal digits.

c) Find the smallest integer n such that n^n contains all 10 decimal digits.

d) Find the smallest integer n such that n! contains one digit 10 times. What is that digit?

PROPOSED PROBLEMS

by

M. Bencze

(i) Solve the following equations:

1) $S^k(x) + S^k(y) = S^k(z)$, $k \in Z$, $x, y, z \in Z$

where S' is the Smarandache function and $S(-n) = -S(n)$

2) $\frac{4}{n} = \frac{1}{S(x)} + \frac{1}{S(y)} + \frac{1}{S(z)}$, $n > 4$

3) $\frac{5}{n} = \frac{1}{S(x)} + \frac{1}{S(y)} + \frac{1}{S(z)}$, $n > 5$

4) $S^{S(y)}(x) = S^{S(x)}(y)$

5) $S\left(\sum_{k=1}^{n} x_k^u\right) = S^u\left(\sum_{k=1}^{n} x_k\right)$, $u \in Z$

6) $S^y(x) - S^t(z) = S^{y-t}(x-z)$

7) $\sum_{k=1}^{n} S^m(x_k) = \sum_{k=n}^{2n} S^m(x_k)$

8) $2S(x^4) - S^2(y) = 1$

9) $S\left(\frac{x+y+z}{3}\right) + \frac{S(x)+S(y)+S(z)}{3} = \frac{2}{3}\left[S\left(\frac{x+y}{2}\right) + S\left(\frac{y+z}{2}\right) + S\left(\frac{z+x}{2}\right)\right]$

10) $S(x_1^{x_1}) \cdot S(x_2^{x_2}) \ldots S(x_n^{x_n}) = S(x_{n+1}^{x_{n+1}})$

11) $S(x_1^{x_2}) \cdot S(x_2^{x_3}) \ldots S(x_{n-1}^{x_n}) = S(x_n^{x_1})$

12) $S(x) = \mu(y)$, where μ is the Möbius function

13) $S^2(Q_n) = \sum_{Q_{n-1} | Q_n} \ldots \sum_{Q_2 | Q_3} \sum_{Q_1 | Q_2} \mu^2(Q_1)$

14) $S(x) = B_y$, where B_y is a Bernoulli number

15) $S(x+y)\,(\,S(x) - S(y)\,) = S(x-y)\,(\,S(x) + S(y)\,)$

16) $S(x) = F_y$, where F_y is a Fibonacci number

17) $\sum_{k=1}^{n} S(k^p) = \sum_{k=1}^{n} S^p(k)$

18) $\sum_{k=1}^{n} S(k) = S\left(\frac{n(n+1)}{2}\right)$

19) $\sum_{k=1}^{n} S(k^2) = S\left(\frac{n(n+1)(2n+1)}{6}\right)$

20) $\sum_{k=1}^{n} S(k^3) = S\left(\frac{n^2(n+1)^2}{4}\right)$

21) $\sum_{k=1}^{n} k(S(k)!) = (S(n+1))! - 1$

22) $\sum_{k=1}^{n} \frac{1}{S(k)S(k+1)} = \frac{S(n)}{S(n+1)}$

(ii) Solve the system

$$\begin{cases} S(x) + S(y) = 2S(z) \\ S(x) \cdot S(y) = S^2(z) \end{cases}$$

(iii) Find n such that n divides the sum

$$1^{S(n-1)} + 2^{S(n-1)} + \ldots (n-1)^{S(n-1)} + 1$$

(iv) May be writen every positive integer n as

$$n = S^3(x) + 2\,S^3(y)\,3\,S^3(z) \quad ?$$

(v) Prove that

$$|S(x) + S(y) + S(z)| + |S(x)| + |S(y)| + |S(z)| \geq$$

$$\geq |S(x) + S(y)| + |S(y) + S(z)| + |S(z) + S(x)|$$

for all $x, y, z \in Z$

(vi) Find all the positive integers x, y, z for which

$$(x+y+z) + S(x) + S(y) + S(z) \geq S(x+y) + S(y+z) + S(z+x)$$

(vii) There exists an infinity of prime numbers which may be writen under the form

$$P = S^3(x) + S^3(y) + S^3(z) + S^3(t) \quad ?$$

(viii) Let $M_1, M_2, \ldots, M_n$ be finite sets and $a_{ij} = \text{card}(M_i \cap M_j)$, $b_{ij} = S(a_{ij})$. Prove that $\det(a_{ij}) \geq 0$ and $\det(b_{ij}) \geq 0$.

(ix) Find the sum

$$\sum_{Q_{n-1}|Q_n} \cdots \sum_{Q_2|Q_3} \sum_{Q_1|Q_2} \frac{1}{S^2(Q_1)}$$

(x) Prove that

$$\sum_{k=1}^{\infty} \frac{1}{S^2(k) - S(k) + 1} \quad \text{is irational}$$

(xi) Find all the positive integers x for which

$$S\left(\left[\frac{x^{n+1} - 1}{(n+1)(x-1)}\right]\right) \geq S\left(\left[x^{\frac{n}{2}}\right]\right)$$

where [x] is the integer part of x.

(xii) There exists at lest a prime between S(n)!, and S(n+1)! ?

(xiii) If $\sigma \in S_n$ is a permutation, prove that

$$\sum_{k=1}^{n} \frac{\sigma(k)}{S^{m+1}(k)} \geq \sum_{k=1}^{n} \frac{1}{k^m}$$

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PROPOSED PROBLEM*

by

I. M. Radu

Show that (except for a finite set of numbers) between S(n) and S(n+1) there exist at least a prime number. (One notes by S(n) the Smarandache Function: the smallest integer such that S(n)! is divisible by n.)

If $N_s(n)$ denotes the number of primes between S(n) and S(n+1), calculate an asymptotic formula for $N_s(n)$.

Some comments:

If n or n+1 is prime, then S(n) or S(n+1) respectively is prime. And the above conjuncture is solved.

But I was not able to find a general proof. It might be a useful thing to apply the Brensch Theorem (if $n \geq 48$, then there exist at least a prime between n and $\frac{9}{8}n$), in stead of Bertrand Postulate / Tchebychev Theorem (between n and $2n$ there exist at least a prime)

The last question may be writing as

$$N_s(n) = |\Pi(S(n+1)) - \Pi(S(n))| ,$$

where $\Pi(x)$ is the number of primes $\leq x$,
but how can we compose the function Π and S ?

References:

[1] Dumitrescu Constantin, "The Smarandache Function", in "Mathematical Spectrum", Sheffield, Vol. 29, No. 2, 1993, pp. 39-40.

[2] Ibstedt Henry, "The F. Smarandache Function S(n): programs, tables, graphs, comments", in "Smarandache Function Journal", Vol. 2-3, No. 1, 1993, pp. 38-71.

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*Charles Ashbacher (U.S.A.), using a computer program that computes the values of S(n) conducted a search up through n<1,033,197 and found where there is no prime p, where $S(n) \leq p \leq S(n+1)$. They are as follows:

n = 224	and S(n) = 8,	S(n+1) = 10
n = 2057	and S(n) = 22,	S(n+1) = 21
n = 265225	and S(n) = 206,	S(n+1) = 202
n = 843637	and S(n) = 302,	S(n+1) = 298

PROPOSED PROBLEMS

by

M. R. Mudge

Problem 1:

The Smarandache no prime digits sequence is defined as follows:
1,4,6,8,9,10,11,1,1,14,1,16,1,18,19,0,1,4,6,8,9,0,1,4,6,8,9,40,41,
42,4,44,4,46,4,48,49,0,...
(Take out all prime digits of n.)
Is it any number that occurs infinetely many times in this sequence?
(for example 1, or 4, or 6, or 11, etc.).

Solution by Dr. Igor Shparlinski,
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It seems that: if, say n has already occured, then for example n3, n33, n333, etc. gives an infinitely many repetitions of this number.

Problem 2:

The Smarandache no square digits sequence is defined as follows:
2,3,5,6,7,8,2,3,5,6,7,8,2,2,22,23,2,25,26,27,28,2,3,3,32,33,3,35,
36,37,38,3,2,3,5,6,7,8,5,5,52,52,5,55,56,57,58,5,6,6,62,...
(Take out all square digits of n.)
Is it any number that occurs infinetely many times in this sequence?
(for example 2, or 3, or 6, or 22, etc. ?)

Solution by Carl Pomerance (E-mail: carl@alpha.math.uga.edu):
If any number appears in the sequence, then clearly it occurs infinitely often, since if the number that appears is k, and it comes from n by deleting square digits, then k also comes from 10n.

Problem 3:

How many regions are formed by joining, with straight chords, n point situated regularly on the circumference of a circle ?

The degeneracy from the maximum possible number of regions for n points on the circumference of a circle seems almost intractable in general.

Perhaps the use of regularly distributed point, i.e. separated by $\frac{2\pi}{n}$ radians, produces the Smarandache Portions of Pi (e) !!

Unsolved Problems

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Welcome to the first installment of what is to be a regular feature in **Smarandache Notions**! In this column, we will present problems where the solution is either unknown or incomplete. This is meant to be an interactive endeavor, so input from readers is strongly encouraged. Always feel free to contact the editor at any of the addresses given above. It is hoped that we can together advance the flow of mathematics in some small way. There will be no deadlines here, and even if a problem is completely solved, new insights or more elegant proofs are always welcome. All correspondents who are the first to resolve any issue appearing here will have their efforts acknowledged in a future column.

While there will almost certainly be an emphasis on problems related to Smarandache notions, it will not be exclusive. Our goal here is to be interesting, challenging and maybe at times even profound. In modern times computers are an integral part of mathematics and this column is no exception. Feel free to include computer programs with your submissions, but please make sure that adequate documentation is included. If you are someone with significant computer resources and would like to be part of a collective effort to resolve outstanding problems, please contact the editor. If such a group can be formed, then sections of a problem can be parceled out and all those who participated will be given credit for the solution.

And now, it is time to stop chatting and get to work!

Definition of the Smarandache function S(n):

S(n) = m, smallest positive integer such that m! is evenly divisible by n.

In [1], T. Yau posed the following question:

For what triplets n, n+1, and n+2 does the Smarandache function satisfy the Fibonacci relationship

$$S(n) + S(n+1) = S(n+2)?$$

And two solutions

S(9) + S(10) = S(11); S(119) + S(120) = S(121)

were given.

In [2], C. Ashbacher listed the additional solutions

S(4900) + S(4901) = S(4902); S(26243) + S(26244) = S(26245);
S(32110) + S(32111) = S(32112) ; S(64008) + S(64009) = S(64010);
S(368138) + S(368139) = S(368140) ; S(415662) + S(415663) = S(415664)

discovered in a computer search up through n = 1,000,000. He then presented arguments to support the conjecture that the number of solutions is in fact infinite.

Recently, Henry Ibstedt from Sweden sent a letter in response to this same problem appearing in the October issue of **Personal Computer World**. He has conducted a more extensive computer search, finding many other solutions. His conclusion was, "This study strongly indicates that the set of solutions is infinite." The complete report has been submitted to **PCW** for publication.

Another problem dealing with the Smarandache function has been given the name Radu's problem, having been first proposed by I.M. Radu[3].

Show that, except for a finite set of numbers, there exist at least one prime number between S(n) and S(n+1).

Ashbacher also dealt with this problem in [2] and conducted another computer search up through n = 1,000,000. Four instances where there are no primes between S(n) and S(n+1) were found.

n = 224 = 2*2*2*2*2*7 S(n) = 8 n+1 = 225 = 3*3*5*5 S(225) = 10
n = 2057 = 11*11*17 S(n) = 22 n+1 = 2058 = 2*3*7*7*7 S(2058) = 21
n = 265225 = 5*5*103*103 S(n) = 206 n+1 = 265226 = 2*13*101*101
S(265226) = 202
n = 843637 = 37*151*151 S(n) = 302 n+1 = 843638 = 2*19*149*149
S(843638) = 298

The fact that the last two solutions involve the pairs of twin primes (101,103) and (149,151) was one point used to justify the conjecture that there is an infinite set of numbers such that there is no prime between S(n) and S(n+1).

Ibstedt also extended the computer search for solutions and found many other cases where there is no prime between S(n) and S(n+1). His conclusion is quoted below.

"A very large set of solutions was obtained. There is no indication that the set would be finite."

This conclusion is also due to appear in a future issue of **Personal Computer World**.

The following statement appears in [4].

$$1141^6 = 74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6$$

This is the smallest known solution for 6th power as the sum of 7 other 6th powers.

Is this indeed the smallest such solution? No one seems to know. The editor would be interested in any information about this problem. Clearly, given enough computer time, it can be resolved. This simple problem is also a prime candidate for a group effort at resolution.

Another related problem that would be also be a prime candidate for a group effort at computer resolution appeared as problem 1223 in **Journal of Recreational Mathematics**.

Find the smallest integer that is the sum of two unequal fifth powers in two different ways, or prove that there is none.

The case of third powers is well known as a result of the famous story concerning the number of a taxicab

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

as related by Hardy[4].

It was once conjectured that there might be a solution for the fifth power case where the sum had about 25 decimal digits, but a computer search for a solution with sum $< 1.02 \times 10^{26}$ yielded no solutions[5].

Problem (24) in [6] involves the Smarandache Pierced Chain(SPC) sequence.

{ 101, 1010101, 10101010101, 101010101010101, . . . }

or

SPC(n) = 101 * 1 0001 0001 . . . 0001
| ---- |

where the section in | ---- | appears n-1 times.

And the question is, how many of the numbers

SPC(n) / 101 are prime?

It is easy to verify that if n is evenly divisible by 3, then the number of 1's in SPC(n) is evenly divisible by 3. Therefore, so is SPC(n). And since 101 is not divisible by 3, it follows that

SPC(n) / 101

must be divisible by 3.

A simple induction proof verifies that SPC(2k)/101 is evenly divisible by 73 for k = 1,2,3, . . .

Basis step:

SPC(2)/101 = 73*137

Inductive step:

Assume that SPC(2k)/101 is evenly divisible by 73. From this, it is obvious that 73 divides SPC(2k). Following the rules of the sequence, SPC(2(k+1)) is formed by appending 01010101 to the end of SPC(2k). Since

01010101 / 73 = 13837

it follows that SPC(2(k+1)) must also be divisible by 73.

Therefore, SPC(2k) is divisible by 73 for all $k > 0$. Since 73 does not divide 101, it follows that SPC(2k) / 101 is also divisible by 73.

Similar reasoning can be used to obtain the companion result.

SPC(3 + 4k) is evenly divisible by 37 for all $k > 0$.

With these restrictions, the first element in the sequence that can possibly be prime when divided by 101 is

SPC(5) = 1010101010101010101.

However, this does not yield a prime as

SPC(5) = 41 * 101 * 271 * 3541 * 9091 * 27961.

Furthermore, since the elements of the sequence SPC(5k), $k > 0$ are made by appending the string

$$0101010101010101010101 = 41 * 101 * 271 * 3541 * 9091 * 27961$$

to the previous element, it is also clear that every number SPC(5k) is evenly divisible by 271 and therefore so is SPC(5k)/101.

Using these results to reduce the field of search, the first one that can possibly be prime is SPC(13)/101. However,

$$SPC(13)/101 = 53 * 79 * 521 * 859 * .....$$

SPC(17)/101 is the next not yet been filtered out. But it is also not prime as

$$SPC(17)/101 = 103 * 4013 * .....$$

The next one to check is SPC(29)/101, which is also not prime as

$$SPC(29)/101 = 59 * 349 * 3191 * 16763 * 38861 * 43037 * 62003 * ....$$

SPC(31)/101 is also not prime as

$$SPC(31)/101 = 2791 * ....$$

At this point we can stop and argue that the numerical evidence strongly indicates that there are no primes in this sequence. The problem is now passed on to the readership to perform additional testing or perhaps come up with a proof that there are no primes in this sequence.

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1. T. Yau: `A Problem Concerning the Fibonacci Series', Smarandache Function Journal, V. 4-5, No. 1, (1994), page 42.

2. C. Ashbacher, **An Introduction To the Smarandache Function**, Erhus University Press, 1995.

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4. D. Wells, **The Penguin Dictionary of Curious and Interesting Numbers**, Penguin Books, 1987.

5. R. Guy, **Unsolved Problems in Number Theory 2nd. Ed.**,Springer Verlag, 1994.

6. F. Smarandache, **Only Problems, Not Solutions!**, Xiquan Publishing House, 1993.

Review of

Have Some Sums To Solve: The Compleat Alphametics Book, by Steven Kahan, Baywood Publishing Company, Amityville, NY, 1978. 114 pp.(paper), $12.45 including poastage, ISBN 0-89503-007-1.

At Last!! Encoded Totals Second Addition, by Steven Kahan, Baywood Publishing Company, Amityville, NY, 1994. 122 pp.,(paper), $12.45 including postage, ISBN 0-89503-171-X.

To many people, alphametics, problems where letters replace digits and those letters form the words of a message, are enjoyable to do, but clearly restricted to the area known as recreational mathematics. However, such an approach is simplistic. Solving a properly constructed alphametic is an exercise in logic and basic number theory that forces the solver to use many elementary rules of arithmetic and algebra if the solution is to be found in a reasonable length of time.

Steven Kahan, the longtime editor of the Alphametics Column of **Journal of Recreational Mathematics**, is clearly the leading expert on this form of problem and these two books present many of his best efforts. The problems and messages are quite good and detailed solutions to all problems are included.

For example, replace the letters of the following message with digits so that the addition is correct

```
  ROMANS
+     ALSO
+     MORE
+       OR
+     LESS
+   ADDED
---------
  LETTERS
```

If you like logic puzzles or are a teacher looking for extra credit problems that involve more complex, yet elementary mathematics, either or both of these books would be an excellent solution to your problem.

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Review of

Circles: A Mathematical View, by Dan Pedoe, The Mathematical Association of America, Washington, D. C., 1995. 144 pp. $18.95(paper). ISBN 0-88385-518-6.

Although it is the simplest of all nonlinear geometric forms, the circle is far from trivial. It is indeed a pleasure that The Mathematical Association of America chose to reprint an update of this classic first printed in 1957. Geometry teaching has been in retreat for many years in the US and that has been a sad (and very bad) thing. It is also puzzling as so many people say that the reason why they cannot do mathematics (i.e. algebra) is that they need to see something in order to understand it. Furthermore, the first mathematical education most children receive contains the differentiation of shapes and their different properties.

Circles and lines as used in geometry are abstractions that are easily grasped, much simpler to many than the abstract generalizations of algebra. One can only hope that this book signals a rebirth in interest in geometry education. Without question, it can be used as a text for that education and would help parent a rebirth. To remedy this modern affliction and make the material available to the current readership, a chapter zero was included. This new chapter is used to introduce the background concepts and terminology that could be assumed when it was first published.

No one can truly appreciate the intellectual achievements of the ancients as summarized by Euclid without doing some of the problems. There is also a stark beauty to a form of mathematics where the tools are a compass, straightedge and a mind. Particularly in the age of calculators and computers. All of the basic, ancient, results concerning circles are covered as well as some very recent ones. The theorems are well presented and complete without being overdone. In keeping with the ancient traditions, pencil, paper, compass and straightedge are the only tools used. A short collection of solved exercises is also included.

Like the books of Euclid, this work will grow old but never dated. It was destined to be a classic when it was first printed and remains so today.

From Erdos to Kiev: Problems of Olympiad Caliber, edited by Ross Honsberger, The Mathematical Association of America, 1995. 250 pp., $31.00(paper). ISBN 0-88385-324-8.

Mathematicians by definition have a love affair with good problems, and this is a collection of the best. While designed to be at a level for mathematical olympiad use, all mathematicians will find something in here that will stretch them. Some are at the level where the solution requires a simple insight, but others may require reaching for your thinking cap. However, all can be solved using arguments considered within the reach of an olympic mathlete. Which is encouraging. It is nice to know that there are young people who can do problems that force me to strain a few neurons. Solutions are included, most of which were created by the editor. The problems are taken from geometry, number theory, probability and combinatorics.

Another high quality entry in the series of problem books by Ross Honsberger, this is a book for all mathematicians, potential olympiads to professionals.

Reviewed by
Charles Ashbacher

Review of

An Introduction to the Smarandache Function, by Charles Ashbacher, Erhus University Press, 1995, 60 pp. (paper), $7.95.
ISBN 1-879585-49-9

This slim volume patently lives up to its title. It does give an introduction to the Smarandache function reaching from its definition all the way to an enumeration and brief discussion of several unsolved problems. Theorems are clearly stated and proofs are always supplied. However, the exposition is relatively lively and informal, lending to this book's readability and brevity. One could get an overview of the topic by skimming this book in an hour or two, skipping the proofs and algorithms. The more diligent reader will spend considerably more time constructing his own examples to illustrate the proofs and test the algorithms.

Chapter one covers basics of the number theoretic Smarandache function, S(n), where n is a positive integer. Included are its definition, 16 theorems and a ready-to-use C++ program for computing values of this function. A background in Number Theory is certainly helpful for approaching this topic, but not absolutely necessary. Just in case, the chapter begins with a one page summary of the idea of divisibility and definitions of the standard arithmetic functions f, s and t. It culminates with a theorem characterizing the range of S(n). The author has considerable experience in computer investigations of this and other topics in number theory and recreational mathematics. In addition to the C++ implementation, he has supplied a UBASIC program, useful for handling extremely large numbers which surpass the maximum allowable integer size of C++.

Chapter two takes up some deeper questions. Topics include iteration and fixed points of the Smarandache function as well as solutions of numerous equations such as the Fibonacci-like relation $S(n+2) = S(n+1) + S(n)$ *. Various problems are presented and solved. Many other, as yet unsolved, problems are presented. In the latter case the author often furnishes a conjecture along with helpful rationale. The reader is led to the jumping off place, ready for his own foray into unresolved areas of investigation. These conjectures and plausibility arguments are clearly labelled as such and hence distinguishable from the theorems and proofs with which they are interspersed.*

This book is not without its niggling errors, mostly typographical and obvious enough as to cause no serious confusion. A few discrepancies in terminology and notation were also noted, probably not uncommon in the literature pertaining to a mathematical topic which is less than 20 years old. As Ashbacher notes in his introductory material, the Smarandache function was created in the 1970's and first published in 1980. In this work, he has given us a bibliography guiding us to works published in the intervening years and provided a good roadmap taking us from the beginnings to the current state of knowledge of his topic.

Reviewed by

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JOURNAL OF RECREATIONAL MATHEMATICS

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